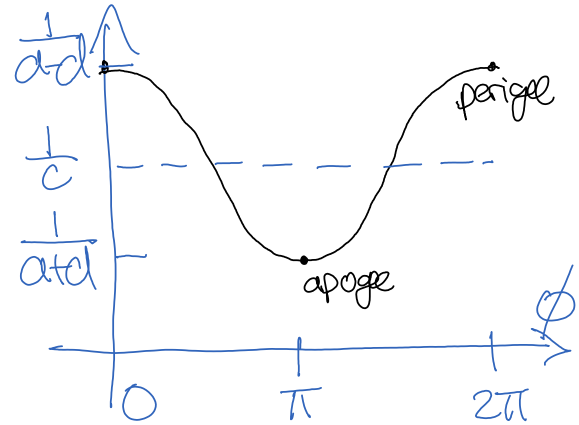
L32 Elliptical orbits

Thursday, November 15, 2018   20:26

3) Equation of an ellipse

$$\frac{1}{\rho} = \frac{1}{c}(1+\varepsilon\cos\phi) \qquad \rho^2 = x^2+y^2$$

$$\rho(1+\varepsilon\cos\phi) = c \qquad x = \rho\cos\phi$$



$$\frac{1}{a-d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{a+d}$$
$$\frac{1}{a-d} + \frac{1}{a+d} = \frac{2a}{a^2-d^2} = \frac{2}{c}$$
$$ac = a^2-d^2$$

$$\sqrt{x^2+y^2} + \varepsilon x = c$$

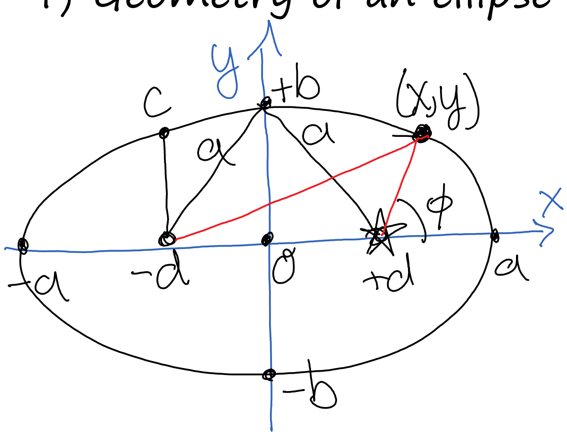
$$x^2+y^2 = (c-\varepsilon x)^2 = \varepsilon^2 x^2 - 2\varepsilon c x + c^2$$

$$(1-\varepsilon^2)x^2 + 2\varepsilon c x + y^2 = c^2$$

$$(1-\varepsilon^2)\left(x + \frac{\varepsilon}{1-\varepsilon^2}c\right)^2 + y^2 = c^2\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right) = c^2\frac{1}{1-\varepsilon^2}$$

$$\left(\frac{x+d}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad a = \frac{c}{1-\varepsilon^2} \quad b = \frac{c}{\sqrt{1-\varepsilon^2}} \quad d = \varepsilon a$$

4) Geometry of an ellipse



$$\boxed{\begin{aligned} a^2 - d^2 &= b^2 = ac \\ &= (a-d)(a+d) \quad \rho_{min}\cdot\rho_{max} \\ d &= \varepsilon a \\ c &= (1-\varepsilon^2)a \\ b &= \sqrt{1-\varepsilon^2}\,a \end{aligned}}$$

(2 free parameters)

- a = semimajor axis
- b = semi minor axis
- c = semi latus rectum
- d = linear eccentricity
- ε = eccentricity

string of length 2a fixed at two foci traces ellipse:

$$\sqrt{(x-d)^2+y^2}+\sqrt{(x+d)^2+y^2}=2a$$

$$(\sqrt{\ })^2=(-\sqrt{\ }+2a)^2=(\sqrt{\ })^2-4a\sqrt{\ }+4a^2$$

square to get rid of $\sqrt{\ }$

$$(x-d)^2+y^2=(x+d)^2+y^2-4a\sqrt{(x+d)^2+y^2}+4a^2$$

$$-4xd-4a^2=-4a\sqrt{(x+d)^2+y^2}$$

isolate final $\sqrt{\ }$ and square once more

$$d^2x^2+2da^2x+a^4=a^2x^2+2a^2dx+a^2d^2+a^2y^2$$

$$(a^2-d^2)x^2+a^2y^2=a^2(a^2-d^2)$$

collect terms.

$$(x/a)^2+(y/b)^2=1 \qquad \underbrace{b^2+d^2=a^2}_{\text{abd triangle (string)}}$$

eq'n of ellipse

if $d=\varepsilon a$ then $b=\sqrt{1-\varepsilon^2}\,a$

solve for c in two different ways:

$$2a=c+\sqrt{c^2+(2d)^2}$$
$$4a^2-4ac+\cancel{c^2}=\cancel{c^2}+4d^2$$
$$a^2-ac=d^2$$
$$c=\frac{a^2-d^2}{a}=a(1-\varepsilon^2)$$

$$(d/a)^2+(c/b)^2=1$$
$$c^2=b^2\left(1-\left(\frac{d}{a}\right)^2\right)$$
$$=(1-\varepsilon^2)a^2(1-\varepsilon^2)$$
$$c=(1-\varepsilon^2)a$$