L40 Euler's equations

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* Review of Rotational Mechanics:

- Rotational dynamics: $\dot{\vec{l}} = \vec{N}$ $\boxed{\vec{l} = \vec{r} \times \vec{p}}$

$$\dot{\vec{l}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \cancel{\dot{\vec{r}} \times \vec{p}} + \vec{r} \times \dot{\vec{p}} = \boxed{\vec{r} \times \vec{F} \equiv \vec{N}} \text{ or } \vec{\tau} \text{ or } \vec{\Gamma}$$

- Inertia: $\vec{l} = \overleftrightarrow{I} \vec{\omega}$ $\overleftrightarrow{I} = m(r^2 I - \vec{r} \otimes \vec{r}) = m(r^T r I - r r^T)$

$$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m(\vec{r} \times \vec{\omega}) = -m\vec{r} \times (\vec{r} \times \vec{\omega}) = m(r^2 - \vec{r}\vec{r}\cdot)\vec{\omega} \equiv \overleftrightarrow{I}\vec{\omega}$$

$$\overleftrightarrow{I} = m(r^T r I - r r^T) = m^{(x\,y\,z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}} I - \begin{pmatrix} x \\ y \\ z \end{pmatrix}(x\,y\,z) = m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

- since "moment of inertia tensor" $I^T = I$, it is diagonalizable
- its eigenvectors/values are called "principle axes/moments" for which $L_i = I_i \omega_i$

- Rotational Energy:

lin: $\vec{p} = m\vec{v}$ $\vec{F} = \dot{\vec{p}} = m\vec{a}$ $\int dW = \vec{F} \cdot d\vec{x} = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} \vec{p} \cdot \vec{v} = \frac{\vec{p}^2}{2m}$

rot: $\vec{l} = \overleftrightarrow{I} \vec{\omega}$ $\vec{N} = \dot{\vec{l}} = \overleftrightarrow{I} \vec{\alpha}$ $\int dW = \vec{N} \cdot d\vec{\theta} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2} \vec{l} \cdot \vec{\omega} = \frac{1}{2} \vec{l} \cdot \overleftrightarrow{I}^{-1} \cdot \vec{l}$

- Summary: all formulas take on the same form, exchanging:

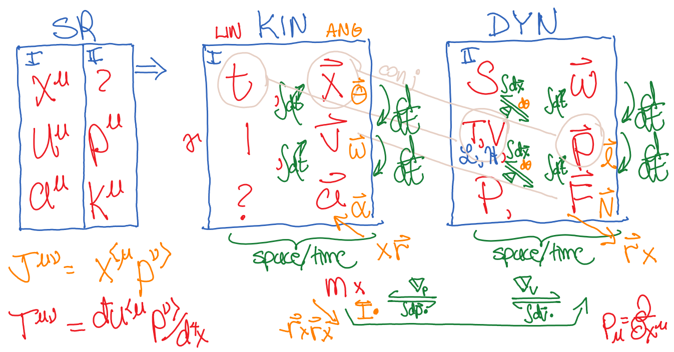
lin:	t	$\vec{r}$ $\vec{v}$ $\vec{a}$	m	$\vec{F}$ $\vec{p}$	T, V
	$\parallel$	$\uparrow \times \vec{r}$	$\downarrow -r_\times r_\times$	$\downarrow \vec{r}_\times$	$\parallel$
rot:	t	$d\vec{\theta}$ $\vec{\omega}$ $\vec{\alpha}$	$\overleftrightarrow{I}$	$\vec{N}$ $\vec{l}$	T, V

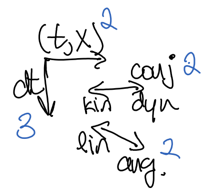
note: $d\vec{\theta}$ not $\vec{\theta}$!

In particular, time and energy are the same;

spatial coordinates : $d\vec{r} = d\vec{\theta} \times \vec{r}$
momentum, force : $\vec{\ell} = \vec{r} \times \vec{p}$ (backwards!)
inertia: : $\overleftrightarrow{I} = -m \cdot \vec{r} \times \vec{r} \times$

- Kinematics/Dynamics Map





* Euler's equations:

$\vec{N} = \dot{\vec{\ell}}$ (space) but $\vec{\ell} = \overleftrightarrow{I}\vec{\omega}$ and $\overleftrightarrow{I}$ is only constant in the body frame:

$\vec{N} = \dot{\vec{\ell}} + \vec{\omega} \times \vec{\ell}$ (body) Euler's equations.

$\overleftrightarrow{I}\dot{\vec{\omega}} = -\vec{\omega} \times \overleftrightarrow{I}\vec{\omega}$ in the absense of external forces.

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2$$

(Euler equations)
Newton's laws
for rigid body

* Free Precession (from Goldstein, Pool, Safko, sec 5.6, p 200)

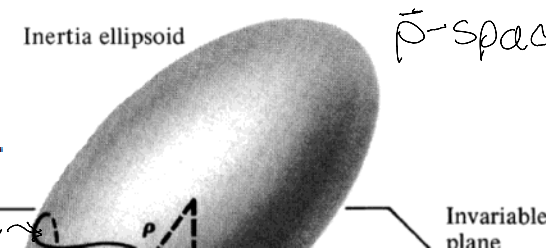
\+ Poinsot construction

$\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = 2T$

$$F(\rho) = \boldsymbol{\rho} \cdot \mathbf{I} \cdot \boldsymbol{\rho} = \rho_i^2 I_i, = 1$$

on the ellipsoid:

$$\boldsymbol{\rho} = \frac{\boldsymbol{\omega}}{\omega\sqrt{I}} = \frac{\boldsymbol{\omega}}{\sqrt{2T}}.$$



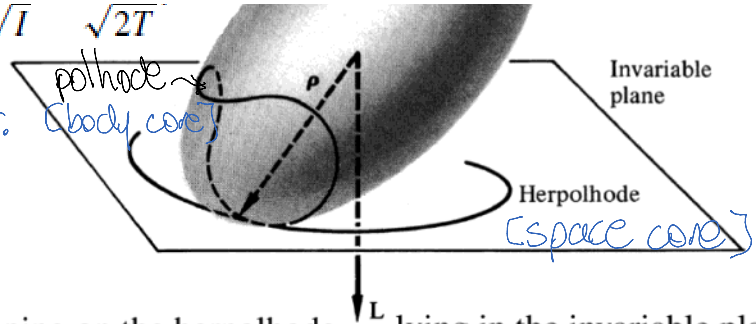
$\vec{\rho}$-space

normal to ellipsoid: $2\mathbf{I} \cdot \boldsymbol{\omega}$ $\sqrt{2}$

$\omega\sqrt{I}$ $\sqrt{2T}$

normal to ellipsoid:
$$\nabla_\rho F = 2\mathbf{I}\cdot\boldsymbol{\rho} = \frac{2\mathbf{I}\cdot\boldsymbol{\omega}}{\sqrt{2T}} = \sqrt{\frac{2}{T}}\mathbf{L}.$$ const.

projection along L:
$$\frac{\boldsymbol{\rho}\cdot\mathbf{L}}{L} = \frac{\boldsymbol{\omega}\cdot\mathbf{L}}{L\sqrt{2T}} = \frac{\sqrt{2T}}{L}$$ const.



the polhode rolls without slipping on the herpolhode lying in the invariable plane.

+ Binet construction $I_3 \le I_2 \le I_1$,

$$T = \frac{L_x^2}{2I_1} + \frac{L_y^2}{2I_2} + \frac{L_z^2}{2I_3}$$ (conserved)

$$\frac{L_x^2 + L_y^2 + L_z^2}{L^2} = 1$$ (conserved)