L41 Rotational Lagrangian

Monday, December 9, 2019   11:00

* Euler's angles (Taylor, sec 10.9, p 401)

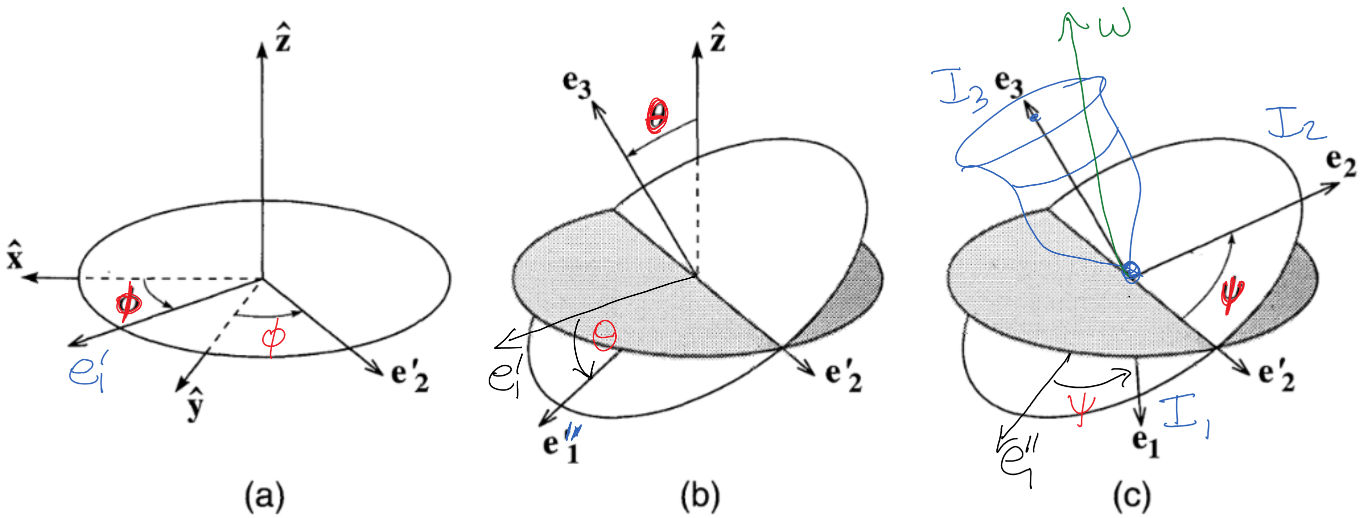

Figure 10.10 Definition of the Euler angles θ, ϕ, and ψ. Starting with the body axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and spaces axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ aligned, the three successive rotations bring the body axes to any prescribed orientation.

\+ calculate the coordinate transformation matrix (rotation)

$$(\hat{e}_1\hat{e}_2\hat{e}_3) = \left[(\hat{e}_1''\hat{e}_2'\hat{e}_3) = \left[(\hat{e}_1'\hat{e}_2'\hat{z}) = (\hat{x}\hat{y}\hat{z})\begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix}\right]\begin{pmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{pmatrix}\right]\begin{pmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{pmatrix}$$

$$= (\hat{x}\hat{y}\hat{z})\begin{pmatrix} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{pmatrix}$$

\+ calculate $\vec{\omega}(\dot{\psi}, \dot{\theta}, \dot{\phi})$ w/ $I_1 = I_2$, I_3 along $\hat{e}_3$

T

$$\dot\phi = \frac{L_z - L_3 c_\theta}{I_1 s_\theta^2}$$

* Lagrangian for spinning top

$$T = \tfrac{1}{2} I_1 (\dot\phi^2 s_\theta^2 + \dot\theta^2) + \tfrac{1}{2} I_3 (\underbrace{\dot\psi + \dot\phi c_\theta}_{\omega_3})^2 \qquad V = MgR c_\theta$$

$$\mathcal{L} = \tfrac{1}{2} I_1 (\dot\phi^2 s_\theta^2 + \dot\theta^2) + \tfrac{1}{2} I_3 (\dot\psi + \dot\phi c_\theta)^2 - MgR c_\theta$$

$$\left(\frac{d}{dt}\frac{\partial}{\partial\dot\theta} - \frac{\partial}{\partial\theta}\right)\mathcal{L} = I_1 \ddot\theta - I_1 \dot\phi^2 s_\theta c_\theta + I_3(\dot\psi + \dot\phi c_\theta)\dot\phi s_\theta - MgR s_\theta = 0$$

$$p_\phi \equiv \frac{\partial\mathcal{L}}{\partial\dot\phi} = I_1 \dot\phi s_\theta^2 + I_3(\dot\psi + \dot\phi c_\theta) c_\theta = I_1\dot\phi s_\theta^2 + L_3 c_\theta = L_z \quad \text{const}$$

$$p_\psi \equiv \frac{\partial\mathcal{L}}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi c_\theta) = I_3\omega_3 = L_3 \quad \text{const} \qquad \boxed{\dot\phi = \frac{L_z - L_3 c_\theta}{I_1 s_\theta^2}}$$

$$E(\theta) = \tfrac{1}{2} I_1 \dot\theta^2 + U_{eff}(\theta) \qquad U_{eff}(\theta) = \frac{(L_z - L_3 c_\theta)^2}{2 I_1 s_\theta^2} + \frac{L_z^2}{2 I_3} + MgR c_\theta$$