Answers to Reading Assignments

Monday, December 3, 2018 12:01

Q01 - Matlab

Q03 - Vectors

Q04 - Vector and Matrix Products

1. Calculate $A\boldsymbol{b}$, A^2, $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}$, or $\boldsymbol{a}^T\boldsymbol{b}$,

where $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$, $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$.

$$A\vec{b} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 2 \\ 5 & 6 & 1 \\ 4 & 4 & 1 \end{pmatrix}$$

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 9 \qquad a^\dagger b = (1\,1\,2)\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 3$$

Q05 - Index Notation

1. evaluate $\delta_{ij}\boldsymbol{a}_i$

$$\delta_{ij} a_j = \delta_{i1} a_1 + \underbrace{\delta_{i2}}_{1 \text{ if } i=2} a_2 + \cdots = a_i$$

2. prove that $\nabla\cdot\nabla\times\boldsymbol{A}(\boldsymbol{r}) = 0$ using the fact that ϵ_{ijk} is totally antisymmetric.

$$\nabla\cdot\nabla\times\vec{A} = \varepsilon_{ijk}\,\partial_i\partial_j A_k = -\varepsilon_{jik}\,\partial_i\partial_j A_k = -\varepsilon_{ijk}\,\partial_i\partial_j A_k$$

$$= -\nabla\cdot\nabla\times\vec{A} \quad \text{if } x=-x \text{ then } 2x=0. \text{ Thus } \nabla\cdot\nabla\times\vec{A}=0$$

3. prove that $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ using the fact that $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$(\vec{a}\times(\vec{b}\times\vec{c}))_k = \varepsilon_{kij}\, a_i\, (b\times c)_j = \varepsilon_{kij}\, a_i\, \varepsilon_{jlm}\, b_l c_m = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\, a_i b_l c_m$$

$$= a_m b_k c_m - a_l b_l c_k = (b\, a\cdot c - c\, a\cdot b)_k$$

Q06 - Complex numbers

1. Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$. What is $z_1 + z_2$? $z_1 z_2$? $z_1^* z_1$?

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

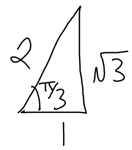
$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + y_1 x_2)$$

$$z_1^* z_1 = (x_1 - i y_1)(x_1 + i y_1) = x_1^2 + y_1^2$$

note: $z_1^* z_2 = (x_1 - i y_1)(x_2 + i y_2) = \underbrace{(x_1 x_2 + y_1 y_2)}_{\vec{p}_1 \cdot \vec{p}_2} + i\,\underbrace{(x_1 y_2 - y_1 x_2)}_{(\vec{p}_1 \times \vec{p}_2)_z}$

2. Write $z = 1 + i\sqrt{3}$ in the form $z = \rho e^{i\theta}$.

$$z = 1 + i\sqrt{3} = 2e^{i\pi/3}$$



3. Given the Taylor expansions $e^s = 1 + s + s^2/2! + s^3/3! + x^4/4! + \ldots$, $\cos(s) = 1 - s^2/2! + s^4/4! - \ldots$, and $\sin(s) = s - s^3/3! + \ldots$ prove Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$

$$e^{i\theta} = 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \cdots$$

$$= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \ldots\right)$$

$$= \cos\theta + i\sin\theta \qquad \text{because } i^2 = -1$$

Q07 - Rotations

1. Rotate the the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ clockwise 45° about the origin.

$$R_{45^\circ}\vec{v} = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2. Show that $R_\theta^T R_\theta = R_{-\theta} R_\theta = I$.

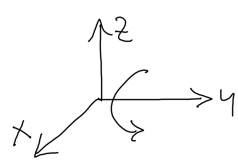
$$R_\theta^T R_\theta = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} = \begin{pmatrix} c_\theta^2 + s_\theta^2 & -c_\theta s_\theta + s_\theta c_\theta \\ -s_\theta c_\theta + c_\theta s_\theta & s_\theta^2 + c_\theta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$(R_\theta^T R_\theta) R_\theta^{-1} = R_\theta^T (\cancel{R_\theta} R_\theta^{-1})^{I} = R_\theta^T = (I) R_\theta^{-1} = R_\theta^{-1} \qquad \text{thus } R_\theta^T = R_\theta^{-1}$$

3. Write the 3-d matrix for a rotation of 90° about the y-axis.

$$R_y(90^\circ) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$\hat{x} \to -\hat{z}$
$\hat{y} \to \hat{y}$
$\hat{z} \to \hat{x}$



Q08 - Stretches

1. Calculate the eigenvalues and at least one eigenvector of the matrix $S = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$.

$$|S - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (3)^2 = 0 \qquad 1 - \lambda = \pm 3$$
$$\lambda = 4, -2$$

$$\lambda = 4: \quad (S - \lambda I)\vec{v} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = 4: \quad (S-\lambda I)\vec{v} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -2: \quad (S-\lambda I)\vec{v} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Q09 – Generalized Coordinates

1. Prove that $\boldsymbol{ds} = \boldsymbol{b}_i dq^i$ using the definition $\boldsymbol{b}_i = \partial \boldsymbol{s}/\partial q^i$

$$ds = \frac{\partial \vec{s}}{\partial q^i} dq^i = \vec{b}_i \, dq^i$$ using the chain rule & $b_i \equiv \frac{\partial \vec{s}}{\partial q^i}$

2. Prove that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$ using definitions $\boldsymbol{b}_i = \partial \boldsymbol{s}/\partial q^i$, $\boldsymbol{b}^j = \nabla q^j$, and the chain rule.

$$\vec{b}_i \cdot \vec{b}^j = \frac{\partial \vec{s}}{\partial q^i} \cdot \nabla q^j = \frac{\partial x}{\partial q^i}\frac{\partial q^j}{\partial x} + \frac{\partial y}{\partial q^i}\frac{\partial q^j}{\partial y} + \dots = \frac{\partial q^j}{\partial q^i} = \delta_i^j$$

3. Prove that $A^i = \boldsymbol{b}^i \cdot \boldsymbol{a}$ using the definition $\boldsymbol{a} = \boldsymbol{b}_i A^i$ and $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$.

$$\vec{b}^i \cdot \vec{a} = \vec{b}^i \cdot (\vec{b}_j A^j) = (\vec{b}^i \cdot \vec{b}_j) A^j = \delta^i_j A^j = A^i$$

4. Prove that $A_i = \boldsymbol{b}_i \cdot \boldsymbol{a}$ using the definition $\boldsymbol{a} = \boldsymbol{b}^i A_i$ and $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$.

$$\vec{b}_i \cdot \vec{a} = \vec{b}_i \cdot (\vec{b}^j A_j) = (\vec{b}_i \cdot \vec{b}^j) A_j = \delta_i^j A_j = A_i$$

Q10 – Metric tensor

1. Prove that $ds^2 = g_{ij} dq^i dq^j$ using $\boldsymbol{ds} = \boldsymbol{b}_i dq^i$ and definition $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$.

$$ds^2 = d\vec{s} \cdot d\vec{s} = (\vec{b}_i dq^i) \cdot (\vec{b}_j dq^j) = (\vec{b}_i \cdot \vec{b}_j) dq^i dq^j = g_{ij} dq^i dq^j$$

2. Prove that $A_i = g_{ij} A^j$ using $A_i = \boldsymbol{b}_i \cdot \boldsymbol{a}$ and definitions $\boldsymbol{a} = \boldsymbol{b}_i A^i$ and $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$.

$$A_i = \vec{b}_i \cdot \vec{a} = \vec{b}_i \cdot (b_j A^j) = (\vec{b}_i \cdot \vec{b}_j) A^j = g_{ij} A^j$$

3. Prove that $\boldsymbol{b}_i = g_{ij} \boldsymbol{b}^j$ by expanding $\boldsymbol{b}_i = (\beta_i)_j \boldsymbol{b}^j$ and using the pattern $A_j = \boldsymbol{b}_j \cdot \boldsymbol{a}$ to find the covariant components $(\beta_i)_j$ of $\boldsymbol{b}_i$. You also need $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$.

$\vec{b}_i = (\beta_i)_j \vec{b}^j$ the components $(\beta_i)_j$ of $\vec{b}_i$ are

$(\beta_i)_j = \vec{b}_j \cdot \vec{b}_i = g_{ij}$ from above (Q09).

4. Prove that $g_{ij} g^{jk} = \delta_i^k$ by combining $A_i = g_{ij} A^j$ with $A^j = g^{jk} A_k$ and using $A_i = \delta_i^k A_k$.

$$A_i = g_{ij} A^j = g_{ij}(g^{jk} A_k) = (g_{ij} g^{jk}) A_k = \delta_i^k A_k$$

thus $g_{ij} g^{jk} = \delta_i^k$

Q11 – Lagrange's equations

1. Prove that $\boldsymbol{\Gamma}_{ij} = \boldsymbol{\Gamma}_{ji}$, where $\boldsymbol{\Gamma}_{ij} \equiv \partial \boldsymbol{b}_i / \partial q^j$ and $\boldsymbol{b}_i = \partial \boldsymbol{s} / \partial q^i$.

$$\vec{\Gamma}_{ij} = \frac{\partial \vec{b}_i}{\partial q^j} = \frac{\partial}{\partial q^j}\frac{\partial \vec{s}}{\partial q^i} = \frac{\partial}{\partial q^i}\frac{\partial \vec{s}}{\partial q^j} = \frac{\partial \vec{b}_j}{\partial q^i} = \vec{\Gamma}_{ji}$$

2. Show that $\boldsymbol{A} = \boldsymbol{b}_i \ddot{q}^i + \boldsymbol{\Gamma}_{ij}\dot{q}^i\dot{q}^j$, starting from $\boldsymbol{A} = \dot{\boldsymbol{v}}$, where $\boldsymbol{v} = \boldsymbol{b}_k \dot{q}^k$, and the dot means derivative with respect to time. Hint: use the product rule and the chain rule $\frac{d\boldsymbol{b}_i}{dt} = \frac{\partial \boldsymbol{b}_j}{\partial q^j}\frac{dq^i}{dt}$.

$$\vec{A} = \dot{\vec{v}} = \frac{d}{dt}(\vec{b}_k \dot{q}^k) = \vec{b}_k \ddot{q}^k + \dot{\vec{b}}_i \dot{q}^i = \vec{b}_k \ddot{q}^k + \left(\frac{\partial \vec{b}_i}{\partial q^j}\dot{q}^j\right)\dot{q}^i$$

$$= \vec{b}_k \ddot{q}^k + \vec{\Gamma}_{ij}\dot{q}^i\dot{q}^j \qquad A^k = \ddot{q}^k + \Gamma_{ij}^k \dot{q}^i \dot{q}^j$$

3. Show that $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^k} = m g_{kj}\ddot{q}^j + m\frac{\partial g_{kj}}{\partial q^i}\dot{q}^i\dot{q}^j$, where $T = \frac{1}{2} m g_{ij}\dot{q}^i\dot{q}^j$.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}^k} T = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^k}\left(\tfrac{1}{2} m g_{ij}\dot{q}^i\dot{q}^j\right) = \frac{d}{dt}\tfrac{1}{2} m g_{ij}\left(\delta^i_k \dot{q}^j + \dot{q}^i \delta^j_k\right)$$

$$= \frac{d}{dt} m g_{kj}\dot{q}^j = m g_{kj}\ddot{q}^j + m\frac{\partial g_{kj}}{\partial q^i}\dot{q}^i\dot{q}^j \quad \text{(chain rule)}$$

Q12 – Lagrange's equations

1. Develop the equation of motion for our poor frog-prince using the Lagrangian.

$$\mathcal{L} = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}(m\dot{x}) = m\ddot{x} = 0 \qquad \ddot{x} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}(m\dot{y}) + mg = m\ddot{y} + mg = 0 \qquad \ddot{y} = -g$$

2. Develop the equation of motion for a single pendulum using the Lagrangian.

$$\mathcal{L} = \tfrac{1}{2} m(l\dot{\theta})^2 - mgl(1 - \cos\theta) \qquad \ddot{\theta} = -\tfrac{g}{l}\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} ml^2\dot{\theta} + mgl\sin\theta = ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Q14 – Linear and Quadratic Drag

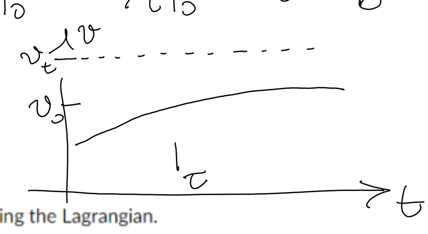
1. Derive $v(t) = v_{ter} + (v_0 - v_{ter})e^{-t/\tau}$ where $v_{ter} = mg/b$ and $\tau = m/b$, for motion in a single direction with net force $F = mg - bv$.

$$m\dot{v} = F = mg - bv \qquad \text{when } F=0,\ v_t = \frac{mg}{b} = g\tau$$

$$\dot{v} = -(v - v_t)/\tau$$

$$\int_{v_0}^{v}\frac{dv}{v - v_t} = \int_0^t -dt/\tau \qquad \ln(v - v_t)\Big|_0^v = -t/\tau\Big|_0^t \qquad \tau = \frac{m}{b}$$

$$v - v_t = (v_0 - v_t)e^{-t/\tau}$$



2. Develop the equation of motion for a single pendulum using the Lagrangian.

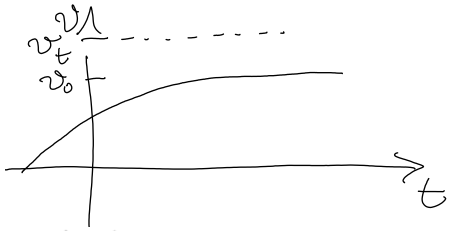
$$m\dot{v} = F = mg - cv^2 \qquad \text{when } F=0,\ v_t^2 = \frac{mg}{c} \qquad \tau^2 \equiv \frac{m}{cg}$$

$$\dot{v} = g(1 - v^2/v_t^2)$$
$$\frac{dv}{1 - v^2/v_t^2} = \frac{v_t}{\tau}\,dt$$
$$\int_{\alpha_0}^{\alpha} d\alpha = \int_0^t dt/\tau$$
$$\alpha - \alpha_0 = t/\tau$$
$$v = v_t \tanh\left(\alpha_0 + \frac{gt}{v_t}\right)$$

$v_t = \tau g \qquad v_t \tau = \frac{m}{c}$

let $v = v_t \tanh\alpha$

$dv = v_t \operatorname{sech}^2\alpha \, d\alpha \qquad 1 - \tanh^2\alpha = \operatorname{sech}^2\alpha$



Q15- Magnetic Fields

1. Using $\eta = v_x + iv_y$ and the magnetic force law $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, show that $\dot{\eta} = -i\omega\eta$ for a particle of mass m and charge q travelling in the magnetic field $\hat{\boldsymbol{z}}B$ so that $\omega = qB/m$.

$$m\dot{\vec{v}} = F = q\vec{v} \times \hat{z}B$$
$$\dot{\vec{v}} = -\frac{qB}{m}\hat{z} \times \vec{v}$$
$$\dot{\eta} = -i\omega\eta$$
where $\omega = \frac{qB}{m}$ is the cyclotron frequency

note: $\hat{z} \times (\hat{x} v_x + \hat{y} v_y) = -\hat{x} v_y + \hat{y} v_x$

$i(v_x + i v_y) = -v_y + i v_x$

thus $\hat{z} \times \vec{v} \sim i\eta$ where $\vec{v} \sim \eta \equiv v_x + i v_y$

2. Integrate $\dot{\eta} = -i\omega\eta$ twice to determine the position $\xi(t) = x + iy$ of the particle which starts from position $C = x_0 + iy_0$ at initial velocity $A = v_{x0} + iv_{y0}$.

$$\frac{d\eta}{\eta} = -i\omega dt \qquad \eta = \eta_0 e^{-i\omega t}$$
$$\xi - \xi_0 = \int_0^t \eta\, dt = \frac{\eta_0}{-i\omega}\left(e^{-i\omega t} - 1\right)$$

L16 - Hamilton's Equations

Q17 - Harmonic Oscillator

1. Solve the equation of motion for the position $x(t)$ of a mass m at the end of a spring of stiffness k with the other end held fixed, given that the mass is held at rest at $t = 0$, but displaced a distance x_0 from equilibrium. You can use either the force $F = -kx$ or potential $V = -\frac{1}{2}kx^2$ of the spring.

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \qquad$ or $\qquad ma = F = -kx$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}m\dot{x} + kx = m\ddot{x} + kx = 0$$

let $x = e^{\alpha t} \qquad (m\alpha^2 + k)x = 0 \qquad \alpha = \pm i\omega \qquad \omega = \sqrt{k/m}$

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t} = \underbrace{C_1 + C_2}_{B_1} \cos\omega t + \underbrace{i(C_1 - C_2)}_{B_2} \sin\omega t$$
$$x_0 = B_1 \cos 0 + B_2 \sin 0 = B_1$$
$$\dot{x}_0 = -\omega B_1 \sin 0 + \omega B_2 \cos 0 = \omega B_2$$
$$x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega} \sin\omega t$$

you can also solve for C_1 & C_2, but B_1, B_2 are easier

Q18 - Impedance Analogy

1. Solve the voltage equation for the charge $Q(t)$ on a capacitor of capacitance C connected to an inductor of inductance L in a series circuit, given that the initial charge on the capacitor is Q_0 and initial current is $I_0 = 0$ at $t = 0$.

$$L\ddot{Q} + \frac{1}{C}Q = 0 \qquad \ddot{Q} + \omega^2 Q = 0 \qquad \omega = \frac{1}{\sqrt{LC}}$$

let $Q = e^{\alpha t}$ $\quad \alpha^2 + \omega^2 = 0 \quad \alpha = \pm i\omega$

$Q = B_1 \cos\omega t + B_2 \sin\omega t$ $\quad$ where $e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$

$Q_0 = Q(0) = B_1 \quad I_0 = \dot{Q}(0) = B_2/\omega$

$Q = Q_0 \cos\omega t + I_0/\omega \sin\omega t$.

Q20 – Damped Oscillations

1. Calculate the damping constants β and ω_0 in terms of m, b, k in the equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ of a damped oscillator, for a mass m attached the end of a spring with force $F = -kx$ with a damping force $F = -b\dot{x}$.

$F = ma = -bv - kx \quad \ddot{x} + b/m\, \dot{x} + k/m\, x = 0 \quad 2\beta = b/m \quad \omega_0^2 = k/m$

2. Calculate the damping constants β and ω_0 in terms of L, R, C for equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ of a tank circuit with an inductor L, resistor R, and capacitor C connected in series.

$\Sigma V = L\dot{I} + IR + Q/C \quad \ddot{Q} + R/L\, \dot{Q} + 1/C\, Q = 0 \quad 2\beta = R/L \quad \omega_0^2 = \frac{1}{LC}$

3. Find the general solution of the equation of damped oscillatory motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$.

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ $\quad$ where $2\beta = b/m \quad \omega_0^2 = k/m$ $\quad$ let $x = e^{\alpha t}$

$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \quad \alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta \pm i\omega_1, \quad \omega_1^2 = \omega_0^2 - \beta^2$

$x = e^{-\beta t}(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t)$

4. Find the critical damping constant β_c as a function of ω_0 in the equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. Above β_c, the motion ceases to oscillate, but immediately damps out.

critical damping: $\beta = \omega_0$ so $\sqrt{\beta^2 - \omega_0^2} = 0$

5. Calculate the coefficients C_1 and C_2 in the solution $x(t) = e^{-\beta t}(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t})$ of $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $\omega_1^2 = \omega_0^2 - \beta^2$, for the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. WARNING: you might want to practice this once or twice!

HINT: you can solve the easier equation $x(t) = e^{-\beta t}(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t)$. This was obtained by expanding $e^{\pm i\omega_1 t} = \cos\omega_1 t \pm i \sin\omega_1 t$ and collecting terms to get $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$.

$x_0 = e^0(B_1 \cos 0 + B_2 \sin 0) = B_1$

$\dot{x}_0 = -\beta e^{-\beta \cdot 0}(B_1 \cos 0 + B_2 \sin 0) + e^{-0}(-\omega_1 B_1 \sin 0 + \omega B_2 \cos 0) = -\beta B_1 + \omega_1 B_2$

$x = e^{-\beta t}(x_0 \cos\omega_1 t + \frac{\dot{x}_0 + \beta x_0}{\omega} \sin\omega_1 t)$

Q21 – Driven Oscillations

1. Show that the above differential operator $D \equiv m\frac{d^2}{dt^2} + b\frac{d}{dt} + k$ is linear, that is: $D[C_1 x_1(t) + C_2 x_t(t)] = C_1 D[x_1(t)] + C_2 D[x_t(t)]$.

$\frac{d}{dt}(C_1 x_1(t) + C_2 x_2(t)) = C_1 \frac{d}{dt} x_1(t) + C_2 \frac{d}{dt} x_2(t)$

also composition and scalar multiplication and addition are linear

$\frac{d^2}{dt^2} C_1 x_1 + C_2 x_2 = \frac{d}{dt}(C_1 \dot{x}_1 + C_2 \dot{x}_2) = C_1 \ddot{x}_1 + C_2 \ddot{x}_2 = C_1 \frac{d^2}{dt^2} x_1(t) + C_2 \frac{d^2}{dt^2} x_2(t)$

2. Show that the solution to $D[x] = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$ (driven damped oscillator) is equal to $x(t) = x_p(t) + x_h(t)$, where $x_p(t)$ is one particular solution to the full differential equation $D[x] = f(t)$, and $x_h(t)$ is the general solution to the homogeneous equation $D[x] = 0$ that we solved last time. Note that the homogeneous solution damps out over time, leaving only the particular solution (the *attractor*).

$D[x] = D(x_p + x_h) = D[x_p] + D[x_h] = f(t) + 0$

if the initial conditions of $x(t)$ are $x_0, \dot{x}_0$

let $x_0^p, \dot{x}_0^p$ be the initial values of $x^p(t)$

then $x_0^h = x_0 - x_0^p$ & $\dot{x}_0^h = \dot{x}_0 - \dot{x}_0^p$ define the initial conditions for the homogeneous equation $D[x^h] = 0$.

3. Show that the particular solution of the damped oscillator $D[x] = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$ driven at pure frequency ω is the real part x of $z = x + iy = Ce^{i\omega t}$, where $C = f_0/(\omega_0^2 - \omega^2 + 2i\beta\omega)$. *Hint:* let f be the real part of $f_0 e^{i\omega t} = f_0(\cos\omega t + i\sin\omega t)$, to obtain the complex equation $D[z] = f_0 e^{i\omega t}$. Substitute $z = Ce^{i\omega t}$ and solve for C.

$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$ where $z = x + iy = Ce^{i\omega t}$ (take real part)

$(-\omega^2 + 2i\beta\omega + \omega_0^2)Ce^{i\omega t} = f_0 e^{i\omega t}$

$$C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega} = Ae^{-i\delta}$$

4. Show that the solution $C = Ae^{-i\delta}$ to question 3 has amplitude $A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + (2\beta\omega)^2}}$ and phase shift $\delta = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega_2}$. *Hint:* take the square root of $A^2 = |C|^2 = CC^*$ to get A, and δ is the polar angle of the complex denominator $\omega_0^2 - \omega^2 + 2i\beta\omega$.

if $z = x + iy$ then $z^* z = x^2 + y^2 = \rho^2$ $\tan\phi = \frac{y}{x}$

thus $A = |C| = \sqrt{C^* C} = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$ $\tan\delta = 2\beta\omega/(\omega_0^2 - \omega^2)$

Q23 - Coupled Oscillators

1. Determine the two coupled equations of motion and therefore the stiffness matrix $\boldsymbol{K}$ and mass matrix $\boldsymbol{M}$ in the equation $\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x}$ for two masses $m_{1,2}$ separating three springs $k_{1,2,3}$ with outside ends fixed, as in Fig. 11.1 .

$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1(x_1) = -(k_1+k_2)x_1 + k_2 x_2$

$m_2\ddot{x}_2 = k_3(-x_2) - k_2(x_2 - x_1) = k_2 x_1 - (k_2+k_3)x_2$

$M\ddot{\vec{x}} = -K\vec{x}$ where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}$ $K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}$

2. Factor out the time dependence $\boldsymbol{x} = \boldsymbol{a}e^{i\omega t}$ of $\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x}$, as usual, to obtain the time-independent equation $(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{a} = \boldsymbol{0}$.

$\vec{x} = \vec{a}e^{i\omega t}$ $\ddot{\vec{x}} = (i\omega)^2\vec{a}e^{i\omega t} = -\omega^2\vec{x}$ $M\omega^2\vec{a} = K\vec{a}$

3. Find the two eigenvalues and eigenvectors of $\boldsymbol{K}$ to solve the above equation $\boldsymbol{K}\boldsymbol{a} = -m\omega^2\boldsymbol{a}$ for the two frequencies and modes of oscillation of the system above in the case of identical masses $m_1 = m_2 = m$ and spring constants $k_1 = k_2 = k_2 = k$.

$|K - \lambda I| = \begin{vmatrix} k_1+k_2-\lambda & -k_2 \\ & \end{vmatrix} = (k_1+k_2-\lambda)(k_2+k_3-\lambda) - k_2^2 = 0$

$$\begin{vmatrix} \cdots & \cdots \\ -k_2 & k_2+k_3-\lambda \end{vmatrix}$$

$$\lambda^2 - (k_1+2k_2+k_3)\lambda + (k_1k_2+k_2k_3+k_3k_4) = \lambda^2 - 2\Sigma\lambda + \Pi = 0$$

$$\lambda_\pm = \Sigma \pm \sqrt{\Sigma^2 - \Pi} \quad \text{thus} \quad \omega_{1,2} = \sqrt{\lambda_\pm / m} \quad \text{where } m_1 = m_2$$

In our case, $k_1 = k_2 = k_3 = k \qquad m_1 = m_2 = m$

so $(2k-\lambda)^2 - k^2 = 0 \qquad \lambda = 2k \pm k = k, 3k$

$\lambda = k: \ \vec{a} = \begin{pmatrix}1\\1\end{pmatrix} \qquad \lambda = 3k: \ \vec{a} = \begin{pmatrix}1\\-1\end{pmatrix}$

Q31 - Conservation of Momentum

1. Show that Newton's third law implies that $\Delta(\vec{p}_1 + \vec{p}_2) = 0$ for the 'internal' interaction between two particles.

NIII $F_{12} = -F_{21}$; thus $\Delta(p_1 + p_2) = F_{12}\Delta t + F_{21}\Delta t = 0$

2. Show that the $\vec{F} = \dot{\vec{P}}$, where $\vec{F}$ is the sum of all forces and $\vec{P}$ is the total momentum.

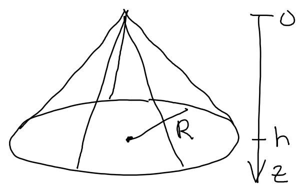
total or just ext.

$$\Sigma \vec{F}_{ext} = \Sigma \vec{F}_{ext} + \underbrace{\Sigma \vec{F}_{int}}_{0} = \Sigma \vec{F} = \Sigma \dot{\vec{p}} = \dot{\vec{P}}$$

3. Show that $\vec{P} = M\dot{\vec{R}}$, where $\vec{P} = \sum \vec{p}_i$ is the total momentum, $M = \sum m_i$, and $\vec{R}$ is the center of mass, ie. $M\vec{R} = \sum m_i \vec{r}_i$.

$$\vec{P} = \Sigma \vec{p}_i = \Sigma m_i \dot{\vec{r}}_i = \frac{d}{dt} \Sigma m_i \vec{r}_i = \frac{d}{dt} M\vec{R} = M\dot{\vec{R}}$$

4. Calculate the center of mass $\vec{R}$ of a cone of radius r and height h.



$$M = \int dm = \int \rho\pi r(z)^2 dz = \int \rho\pi \left(z\frac{R}{h}\right)^2 dz$$
$$= \rho\pi \frac{R^2}{h^2} \int_0^h z^2 dz = \rho \cdot \frac{1}{3}\pi R^2 h$$

$$MZ = \int z\,dm = \rho\pi \frac{R^2}{h^2} \int_0^h z^3 dz = \rho \cdot \frac{1}{4}\pi R^2 h^2$$
$$\vec{R} = (0, 0, 3/4\, h)$$

Q32 - Conservation of Angular Momentum

1. Starting from Newton's second law $\boldsymbol{F} = \dot{\boldsymbol{p}}$ and the definitions of angular momentum $\boldsymbol{\ell} \equiv \boldsymbol{r} \times \boldsymbol{p}$ and torque $\boldsymbol{\Gamma} \equiv \boldsymbol{r} \times \boldsymbol{F}$ about the origin, prove the equivalent law for angular motion $\boldsymbol{\Gamma} = \dot{\boldsymbol{\ell}}$. Show that $\boldsymbol{\ell} = I\boldsymbol{\omega}$, where $I = mr^2$ is moment of inertial for a single particle and $\boldsymbol{\omega} = \hat{\boldsymbol{r}} \times \boldsymbol{v}/r$ is the angular velocity about the origin.

$$\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \underbrace{\dot{\vec{r}} \times \underbrace{\vec{p}}_{m\dot{\vec{r}}}}_{0} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{N}$$

$$\vec{\ell} = \vec{r} \times m\vec{v} = \vec{r} \times m(\vec{\omega} \times \vec{r}) = m(-\vec{r} \times (\vec{r} \times \vec{\omega})) = \overleftrightarrow{I}\vec{\ell}$$

$$\text{where } \overleftrightarrow{I} = -m\vec{r}\times\vec{r}\times = m(r^2 - \vec{r}\vec{r}\cdot) = \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}$$

2. Show that $\boldsymbol{\Gamma} = \boldsymbol{0}$ for a central force $\boldsymbol{F} = \hat{\boldsymbol{r}}F$ and thus $\boldsymbol{\ell}$ is conserved. Use this to prove Kepler's second law, that the line drawn from a planet to the sun sweeps out constant area $d\boldsymbol{A} = \frac{1}{2}\boldsymbol{r} \times d\boldsymbol{r} = \boldsymbol{\ell}dt/2m$ per time dt.

$$\vec{T} = \vec{r} \times \vec{F} = \vec{r} \times \hat{r} F = 0$$
$$d\vec{A} = \frac{1}{2} r \times d\vec{r} = \frac{1}{2m} \vec{r} \times \vec{p}\, dt = \frac{\vec{\ell}\, dt}{2m}$$

$$d\vec{A} = \frac{1}{2} r \times d\vec{r} = \frac{1}{2m} \vec{r} \times \vec{p}\, dt = \frac{\vec{l} dt}{2m}$$

3. [Bonus:] Show that for a system of particles of mass and position $m_i, \boldsymbol{r}_i$, the total angular momentum $\boldsymbol{\ell} = \sum \boldsymbol{\ell}_i$ separates into the sum the angular momentum of the center of mass $\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P}$ and the angular momentum relative to the center of mass $\boldsymbol{\ell}' = \sum (\boldsymbol{\ell}'_i = \boldsymbol{r}'_i \times \boldsymbol{p}'_i)$, using $\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{r}'_i$.

$$\vec{l} = \Sigma \vec{r} \times \vec{p} = \Sigma (\vec{R} + \vec{r}') \times m(\dot{\vec{R}} + \dot{\vec{r}}')$$
$$= \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \cancel{\Sigma m \dot{\vec{r}}'} + \cancel{\Sigma m \vec{r}'} \times \dot{\vec{R}} + \Sigma \vec{r}' \times m\dot{\vec{r}}'$$
$$= \vec{R} \times \vec{P} + \Sigma \vec{r}_i' \times \vec{p}_i' = \vec{R} \times \vec{P} + \Sigma \vec{l}_i'$$

Q34 - Conservation of Energy

1. Use $\boldsymbol{F} = m\boldsymbol{a}$ to show that the net work $W = \int \boldsymbol{F} \cdot d\boldsymbol{r}$ done on an object equals its change of kinetic energy $T = \frac{1}{2} m \boldsymbol{v} \cdot \boldsymbol{v}$.

$$dW = \vec{F} \cdot d\vec{r} = m \frac{d\vec{v}}{dt} \cdot \vec{r} = m\, d\vec{v} \cdot \frac{d\vec{r}}{dt} = m\, d\vec{v} \cdot \vec{v} = d\tfrac{1}{2} m v^2 = dT$$

2. A conservative force $\boldsymbol{F}(\boldsymbol{r}, t)$ is one where $\nabla \times \boldsymbol{F} = \boldsymbol{0}$ and $\partial \boldsymbol{F}/\partial t = 0$. Using Stokes' theorem, show that the integral $V(\boldsymbol{r}) = -\int_0^r \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ is independent of the path of integration. Using the Fundamental Theorem of Vector Calculus, show that the force can be written as the gradient $\boldsymbol{F} = -\nabla V$ of the potential defined above. (see Mason)

$$\int_{0, P_1}^{\vec{r}} \vec{F} \cdot d\vec{r} - \int_{0, P_2}^{\vec{r}} \vec{F} \cdot d\vec{r} = \oint_{P_1 - P_2} \vec{F} \cdot d\vec{r} = \int \nabla \times \vec{F} \cdot d\vec{a} = 0$$

$$\text{thus, } V(\vec{r}) = -\int_0^r \vec{F} \cdot d\vec{r} = -\int_0^r -\nabla V \cdot d\vec{r} = \int_0^r dV$$
$$\text{so } \vec{F} = -\nabla V$$

3. Questions 1 and 2 imply conservation of energy $E = T + V$. Solve this formula for v and integrate $dt = dx/v$ to show that the time taken for a body to go from x_0 to x in one dimension is $t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{(2/m)[E - V(x)]}}$.

$$E = T + V(x) \qquad T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \qquad \int_{t_0}^{t} dt = \frac{dx}{\sqrt{2/m\, T}} = \int_{x_0}^{x} \frac{dx}{\sqrt{2/m\,(E - V(x))}}$$

L35 - Keplerian motion

Q36 - Anatomy of the Inverse Square Law

1. Integrate $V = -\int_\infty^r \boldsymbol{F} \cdot d\boldsymbol{r}$ to find the potential $V(\boldsymbol{r})$ of the unit inverse square central force $\boldsymbol{F} = \hat{\boldsymbol{r}}/4\pi r^2$.

$$V = -\int_\infty^r \vec{F} \cdot d\vec{r} = -\int_{r=\infty}^{r} \frac{\hat{r}}{4\pi r^2} \cdot d\vec{r} = \int_\infty^r \frac{-dr}{4\pi r^2} = \frac{1}{4\pi r}\Big|_\infty^r = \frac{1}{4\pi r}$$

2. Calculate the gradient $\boldsymbol{F} = -\nabla V(\boldsymbol{r})$ of the potential $V(\boldsymbol{r}) = 1/4\pi r$ of an inverse square force.

$$-\nabla \frac{1}{4\pi r} = -\left(\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r S_\theta} \frac{\partial}{\partial \phi}\right) \frac{1}{4\pi r} = \frac{\hat{r}}{4\pi r^2}$$

3. Calculate the curl $\nabla \times \boldsymbol{F}(\boldsymbol{r})$ of the force $\boldsymbol{F} = \hat{\boldsymbol{r}}/4\pi r^2$ to show that it is conservative, and thus has a well-defined potential $V(\boldsymbol{r})$. [if you do it in Cartesian coordinates, one component will suffice]

$$\nabla \times \frac{\hat{r}}{4\pi r^2} = \frac{1}{r^2 S_\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r S_\theta \hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{1}{4\pi r^2} & 0 & 0 \end{vmatrix} = 0$$

4. [bonus] Show that the divergence $\nabla \cdot \boldsymbol{F}(\boldsymbol{r})$ for the force $\boldsymbol{F} = \hat{\boldsymbol{r}}/4\pi r^2$ is zero everywhere but the origin, where it is infinite. Calculate $\oint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{a}$ around the surface of a sphere, so that the divergence theorem $\oint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{a} = \int_V \nabla \cdot \boldsymbol{F}\, d\tau$ becomes $\int_V d\tau \nabla \cdot \boldsymbol{F}(\boldsymbol{r}) = 1$, and we

4. [bonus] Show that the divergence $\nabla\cdot\boldsymbol{F}(\boldsymbol{r})$ for the force $\boldsymbol{F} = \hat{\boldsymbol{r}}/4\pi r^2$ is zero everywhere but the origin, where it is infinite. Calculate $\oint_{\partial V}\boldsymbol{F}\cdot d\boldsymbol{a}$ around the surface of a sphere, so that the divergence theorem $\oint_{\partial V}\boldsymbol{F}\cdot d\boldsymbol{a} = \int_V \nabla\cdot\boldsymbol{F}\,d\tau$ becomes $\int_V d\tau\,\nabla\cdot\boldsymbol{F}(\boldsymbol{r}) = 1$, and we can say $\nabla\cdot\boldsymbol{F}(\boldsymbol{r}) = \delta^3(\boldsymbol{r})$.

$$\nabla\cdot\frac{\hat r}{4\pi r^2} = \frac{1}{\cancel{r^2 s\theta}}\frac{\partial}{\partial r}\cancel{r^2 s\theta}\frac{1}{\cancel{4\pi r^2}} = 0 \text{ except } \infty \text{ @ } r=0.$$

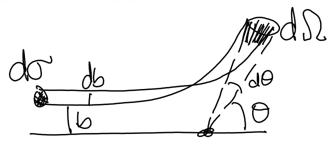
$$\int_V \nabla\cdot\frac{\hat r}{4\pi r^2} = \int_{\partial V} d\vec a\cdot\frac{\hat r}{4\pi r^2} = \int_{\partial V} r^2\,d\Omega\,\frac{1}{4\pi r^2} = \frac{1}{4\pi}\int_{4\pi} d\Omega = 1$$

Thus the total divergence of 1 at $\vec r = 0$: $\nabla\cdot\frac{\hat r}{4\pi r^2} = \delta^3(\vec r)$

Q38 - Cross Section

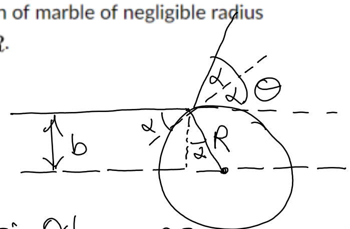
1. Given the relation $b(\theta)$ between scattering angle θ and impact parameter b (we used the symbol s in class), show that the differential cross section is $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$.

$$\frac{d\sigma}{d\Omega} = \frac{b\,db\,\cancel{d\phi}}{s\theta\,d\theta\,\cancel{d\phi}} = \frac{b}{s\theta}\left|\frac{db}{d\theta}\right|$$



2. Calculate differential scattering cross section of marble of negligible radius bouncing off of a fixed bowling ball of radius R.

$$b = R\cos\alpha = R\cos\frac{\theta}{2}$$



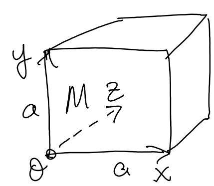
$$\frac{d\sigma}{d\Omega} = \frac{b|db|}{s\theta|d\theta|} = \frac{R^2\cos\frac{\theta}{2}\cdot\left|\frac{-1}{2}\sin\frac{\theta}{2}\right|}{\sin\theta} = \frac{R^2}{4}$$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Bonus: $\sigma = \int\frac{d\sigma}{d\Omega}d\Omega = \int_{4\pi}\frac{R^2}{4}d\Omega = \pi R^2$ (cross-sectional area)

Q41 - Inertia Tensor

1. Calculate I_{xx} about the corner of a cube.



$$I_{xx} = \int dm(y^2+z^2) = \frac{M}{a^3}\int_0^a dx\int_0^a dy\int_0^a dz\,(y^2+z^2)$$

$$= 2\frac{M}{a^3}\int_0^a dx\int_0^a y^2\,dy\int_0^a dz = 2\frac{M}{a^3}\cdot a\cdot\frac{a^3}{3}\cdot a = \frac{2}{3}Ma^2$$

2. Calculate I_{xy} about the corner of a cube.

$$I_{xy} = \int dm(-xy) = \frac{-M}{a^3}\int_0^a x\,dx\int_0^a y\,dy\int_0^a dz = \frac{-M}{a^3}\cdot\frac{a^2}{2}\cdot\frac{a^2}{2}\cdot a = \frac{-1}{4}Ma^2$$

3. Calculate one of the three principal moments of inertia and the corresponding principal axis about the corner of a cube, $I = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$.

$$\overleftrightarrow{I} = \frac{Ma^2}{12}\left\{11\,I - 3\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right\}.$$ Find eigenstuff of $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$!

$$\begin{vmatrix}1-\lambda & 1 & 1 \\ \end{vmatrix} = (1-\lambda)^3 + 2\cdot 1^3 - 3(1-\lambda) = (-\lambda^3+3\lambda^2-3\lambda+1)+2-3+3\lambda$$

$I = \frac{Ma^2}{12}\{11\, I - 3\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\}$. Find eigenstuff of $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$!

$$\begin{vmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{vmatrix} = (1-\lambda)^3 + 2\cdot 1^3 - 3(1-\lambda) = (-\lambda^3+3\lambda^2-3\lambda+1)+2-3+3\lambda$$

$$= -\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda) = 0 \qquad \lambda = 3, 0, 0$$

by symmetry, $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\end{pmatrix}$, thus for $\lambda_1 = 3$ $\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

the orthogonal plane is the eigenspace for $\lambda_{2,3} = 0$.

thus $I_1 = \frac{Ma^2}{12}(11 - 3\cdot 3) = \frac{1}{6}Ma^2$, $I_2 = I_3 = \frac{11}{12}Ma^2$

* Alternative: if you like to think big, calculate directly:

$$\begin{vmatrix}8-\lambda & -3 & -3\\ -3 & 8-\lambda & -3\\ -3 & -3 & 8-\lambda\end{vmatrix} = (8-\lambda)^3 - 2\cdot 3^3 - 3\cdot 3^2(8-\lambda)$$

$$= -\lambda^3 + 3\cdot 8\,\lambda^2 + (-3\cdot 8^2 + 3^3)\lambda + 8^3 - 2\cdot 3^3 - 3^2\cdot 8$$

$$= -\lambda^3 + 24\lambda^2 - 165\lambda + 242$$

$$= -(\lambda-2)(\lambda^2 - 22\lambda + 121) = -(\lambda-2)(\lambda-11)^2 = 0$$

$\lambda = -2, 11, 11$ same as above.