Equivalence of Electrostatic Formulations

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I. Coulomb eq. & Superposition

$$\vec{E} = \int_{R'} \frac{dq'\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$$

Helmholtz

$$\vec{F} = q\vec{E} \qquad W = qV$$

VI?

II. Integral field eq's

$$\Phi_E = Q_{enc.}/\epsilon_0$$
$$\mathcal{E}_E = 0 \quad \text{(closed regions)}$$

Gauss $\rightleftarrows$

Stokes $\rightleftarrows$

III. Differential field eq's

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$
$$\nabla\times\vec{E} = \mathbf{0}$$

Poincaré

$$\mathcal{E}_E = -\Delta V_{\text{(open)}} \quad \overset{\text{FTVC}}{\rightleftarrows} \quad \vec{E} = -\nabla V$$

IV. Potential

$$V = \int \frac{dq'}{4\pi\epsilon_0 \mathscr{r}}$$

Laplace $\rightleftarrows$ Green.

V. Poisson eq.

$$\nabla^2 V = -\rho/\epsilon_0$$

$\Phi_E \equiv \int_S \vec{E}\cdot d\vec{a}$ (flux) is defined through either an open or closed surface S.

$\mathcal{E}_E \equiv \int_P \vec{E}\cdot d\vec{l}$ (flow) is defined along either a closed path [circulation] or an open path P.

$$V \xrightarrow{-\nabla} \vec{E} \xrightarrow{\nabla\cdot} \rho/\epsilon \qquad \left(V \xrightarrow{-\nabla^2} \rho/\epsilon\right)$$

$$\frac{q}{4\pi\epsilon_0 \mathscr{r}} \longrightarrow \frac{q\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} \longrightarrow q\,\delta^3(\vec{\mathscr{r}})$$

Point potential, field, and charge:

$$\nabla_{\mathscr{r}} \frac{1}{\mathscr{r}} = \left(\hat{\mathscr{r}}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{\mathscr{r}}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{\mathscr{r}S_\theta}\frac{\partial}{\partial\phi}\right)\frac{1}{\mathscr{r}} = \frac{-1}{\mathscr{r}^2}$$

$$\nabla_{\mathscr{r}}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \left(\frac{1}{\mathscr{r}^2}\frac{\partial}{\partial\mathscr{r}}\mathscr{r}^2\right)\frac{1}{\mathscr{r}^2} + (\cdots)0 + (\cdots)0 = 0$$

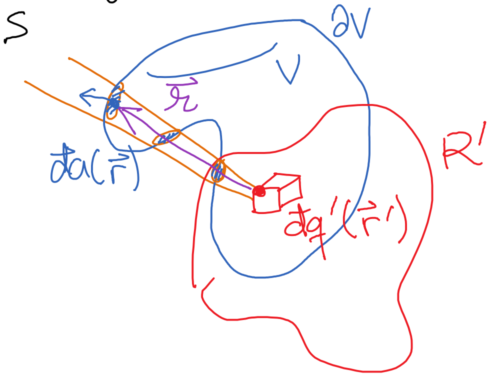
However this is singular (infinite) as $\mathscr{r}\to 0 \;\to\; \delta(\vec{\mathscr{r}})$.
To find the strength (interated divergence) at $\vec{\mathscr{r}} = \vec{0}$,
integrate over a small sphere using the divergence theorem:

$$\int_{\mathscr{r}<\varepsilon} \nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau = \oint_{\mathscr{r}=\varepsilon} d\vec{a}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \oint \hat{\mathscr{r}}\,\cancel{\mathscr{r}^2}\,d\Omega\cdot\frac{\hat{\mathscr{r}}}{\cancel{\mathscr{r}^2}} = \oint_{4\pi} d\Omega = 4\pi$$

thus $\nabla_{\mathscr{r}}\cdot\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta^3(\vec{\mathscr{r}})$

$\mathrm{I} \Rightarrow \mathrm{II}$:

for any Gaussian surface ∂V enclosing a volume V and charge distribution dq' over a region R', the total electrix flux exiting ∂V is



$$\Phi_E = \oint_{\partial V} \vec{E} \cdot \vec{da} = \oint_{\partial V} \int_{R'} \frac{dq' \, \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2} \cdot \vec{da}$$

$$\int_{R'} \frac{dq'}{4\pi\varepsilon_0} \oint_{\partial V} \left[\frac{\hat{\mathscr{r}} \cdot \vec{da}}{\mathscr{r}^2} = d\Omega \right]$$

if dq' is outside ∂V then all $d\Omega$'s spanned by one cone of $d\theta, d\phi$ cancel out.
if dq' is inside ∂V then exactly one $d\Omega$ doesn't cancel.

$$\text{so } \Phi_E = \int_{R'} \frac{dq'}{4\pi\varepsilon_0} \oint_{\partial V} \begin{cases} d\Omega & \text{if } \vec{r}\,' \text{ is inside } \partial V \\ 0 & \text{outside.} \end{cases}$$

note that $\oint_{\text{sphere}} d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi$ let $x = \cos\theta$, $dx = -\sin\theta \, d\theta$

$$= \int_{-1}^{1} dx \cdot \int_0^{2\pi} d\phi = 2 \cdot 2\pi = 4\pi$$

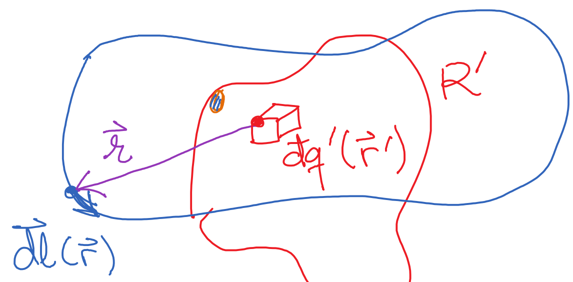
so $\Phi_E = \int_V \frac{dq'}{\varepsilon_0} = Q_{enc}$, since only charges enclosed inside V contribute $\Omega = 4\pi$

thus $\boxed{\Phi_E = Q_{enc}}$

The circulation or flow around any closed path P is

$$\mathcal{E}_E = \oint_P \vec{E}\cdot\vec{d\ell} = \oint_P \int_{R'} \frac{dq\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\cdot\vec{d\ell}$$

$$= \int_{R'} \frac{dq}{4\pi\epsilon_0} \oint_P \frac{\hat{\mathscr{r}}\cdot\vec{d\ell}}{\mathscr{r}^2}$$



$$= \hat{\mathscr{r}}\,d\mathscr{r} + \hat{\theta}\,\mathscr{r}\,d\theta + \hat{\phi}\,\mathscr{r}\sin\theta\,d\phi$$

now $\hat{\mathscr{r}}\cdot\vec{d\ell} = d\mathscr{r}$

and $\frac{d\mathscr{r}}{\mathscr{r}^2} = d\left(\frac{-1}{\mathscr{r}}\right)$ is an exact differential (gradient)
(it only depends on $\mathscr{r}$)

by the FTVC, $\oint_P \frac{\hat{\mathscr{r}}\cdot\vec{d\ell}}{\mathscr{r}^2} = \oint_P \nabla\left(\frac{-1}{\mathscr{r}}\right)\cdot\vec{d\ell} = \oint d\frac{-1}{\mathscr{r}} = \left.\frac{-1}{\mathscr{r}}\right|_a^b = 0$

since P is a closed path, so $a=b$.

thus $\boxed{\mathcal{E}_E = 0}$ around a closed path

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I ⇒ III

$$\nabla\cdot\vec{E} = \nabla\cdot\int_{R'}\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2} = \int_{R'}\frac{dq'}{4\pi\varepsilon_0}\nabla_{\mathscr{r}}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \int_{R'}\frac{d\tau'\,\rho(\vec{r}')}{\cancel{4\pi}\,\varepsilon_0}\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}') = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\nabla\times\vec{E} = \nabla\times\int_{R'}\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2} = \int_{R'}\frac{dq'}{4\pi\varepsilon_0}\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{0}$$

curl: $\frac{1}{\mathscr{r}^2 s_\theta}\begin{vmatrix}\hat{\mathscr{r}} & \frac{1}{\mathscr{r}}\hat{\theta} & \frac{1}{\mathscr{r}s_\theta}\hat{\phi}\\ \partial_{\mathscr{r}} & \partial_\theta & \partial_\phi \\ \frac{1}{\mathscr{r}^2} & 0 & 0\end{vmatrix} = 0$

II ⇔ III

Divergence theorem: for any Gaussian surface S [closed], which bounds a volume V, so that $S = \partial V$,

$$\Phi_E = \oint_{\partial V}\vec{E}\cdot d\vec{a} = \int_V d\tau\,\nabla\cdot\vec{E} \quad\text{and}\quad Q_{enc} = \int_V dq = \int_V d\tau\,\rho(\vec{r})$$

thus $\Phi_E = \frac{Q_{enc}}{\varepsilon_0} \iff \int_V d\tau\,\nabla\cdot\vec{E} = \int_V d\tau\,\rho/\varepsilon_0 \iff \nabla\cdot\vec{E} = \rho/\varepsilon_0$

since this is true for any volume V.

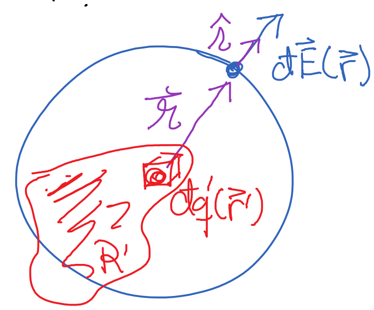
Stokes' theorem: for any Amperian loop [closed path] P which bounds a surface S, so that $P = \partial S$,

$$\mathcal{E}_E = \oint_{\partial S}\vec{E}\cdot d\vec{l} = \int_S d\vec{a}\cdot\nabla\times\vec{E} \quad\text{thus}\quad \mathcal{E}_E = 0 \iff \nabla\times\vec{E} = 0$$

II ⇒ I

Apply the superposition principle the field of each point charge dq', calculated individually using Gauss' law.
Note: $\mathcal{E}_E = 0$ ensures the symmetry of $\vec{E}$ needed for Gauss' law.

for charge dq' at point $\vec{r}'$, calculate the field $d\vec{E}(\vec{r})$ using a sphere of radius $\mathscr{r}$, where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$.
By symmetry, the field is along $\hat{\mathscr{r}}$



thus $\Phi_{dE} = \oint_{\mathscr{r}\text{ const}} d\vec{E} \cdot \hat{\mathscr{r}}\, \mathscr{r}^2 d\Omega = dE_{\mathscr{r}} \cdot 4\pi \mathscr{r}^2 = \frac{dq'}{\varepsilon_0}$

and $d\vec{E} = \frac{dq' \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2}$. Integrating over the charge distribution dq' in R', $\vec{E} = \int_{R'} d\vec{E} = \int_{R'} \frac{dq' \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2}$.

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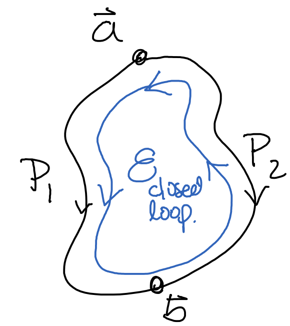
II $\Rightarrow$ IV

$$\mathcal{E}_E = -\Delta V \quad \text{or} \quad -\big(V(\vec b) - V(\vec a)\big) = \int_{\vec a \to \vec b} \vec E \cdot d\vec l$$

is the statement that we can define a potential function $V(\vec r)$ everywhere such that flow $\mathcal{E}_E = \int \vec E \cdot d\vec l$ between any two points $\vec a$ and $\vec b$ is just the potential difference $-\big(V(\vec b) - V(\vec a)\big)$ between those points.



In fact, just define $V(\vec r) = -\int_{\perp}^{\vec r} \vec E \cdot d\vec l$, integrated from a specifically chosen point "$\perp$" called "ground", where the potential is defined to be zero: $V(\perp) = 0$, to the field point $\vec r$.

The fact that $\mathcal{E}_E = 0$ around any closed loop ensures that $V(\vec r)$ is well defined, in that it does not depend on the path between $\perp \to \vec r$. Given two paths P_1 and P_2 from $\perp$ to $\vec r$, Integrate around the loop formed between them:

$$\mathcal{E}_E = \oint \vec E \cdot d\vec l = \int_{P_1} \vec E \cdot d\vec l + \int_{-P_2} \vec E \cdot d\vec l = 0$$

thus $\mathcal{E}_{P_1} - \mathcal{E}_{P_2} = 0$ and $V_1(\vec r) = \int_{P_1} \vec E \cdot d\vec l = \int_{P_2} \vec E \cdot d\vec l = V_2(\vec r)$

Conversely, if the flow along any path $P: \vec{a} \to \vec{b}$ is just the potential difference $\mathcal{E} = -(V(\vec{b}) - V(\vec{a}))$ then the circulation (flow around a closed path) from $\vec{a}$ to $\vec{a}$ is

$$\mathcal{E}_E = -(V(\vec{a}) - V(\vec{a})) = 0$$

Now we have defined V, we can calculate it using Gauss' law → Coulomb's law (II → I above), using ground at $r \to \infty$ (in an arbitrary direction).

$$V(\vec{r}) = -\int_\infty^{\vec{r}} \vec{E} \cdot d\vec{l} = -\int_\infty^{\vec{r}} \int_{R'} \frac{dq\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} \cdot d\vec{l} = \int_{R'} \frac{dq}{4\pi\epsilon_0} \int_\infty^{\vec{r}} \frac{-\hat{\mathscr{r}} \cdot d\vec{l}}{\mathscr{r}^2}$$

We already did this integral in (I → II) noting that $-\frac{\hat{\mathscr{r}} \cdot d\vec{l}}{\mathscr{r}^2} = \nabla \frac{1}{\mathscr{r}} \cdot d\vec{l} = d\frac{1}{\mathscr{r}}$ is a perfect differential,

and by the FTVC, $\int_\infty^{\vec{r}} \frac{-\hat{\mathscr{r}} \cdot d\vec{l}}{\mathscr{r}^2} = \int_\infty^{\vec{r}} \nabla \frac{1}{\mathscr{r}} \cdot d\vec{l} = \int_\infty^{\vec{r}} d\frac{1}{\mathscr{r}} = \left.\frac{1}{\mathscr{r}}\right|_\infty^{\vec{r}} = \frac{1}{\mathscr{r}}$

Thus $V(\vec{r}) = \int_{R'} \frac{dq'}{4\pi\epsilon_0 \mathscr{r}}$

$\underline{\mathrm{I} \Rightarrow \mathrm{IV}}$

We just did this in $\mathrm{II} \to \mathrm{IV}$. To sumarize,

$$\vec{E} = \int_{R'} \frac{dq' \, \hat{r}}{4\pi\epsilon_0 r^2} = \int_{R'} \frac{dq'}{4\pi\epsilon_0} \left(-\nabla_r \frac{1}{r}\right) = -\nabla \int_{R'} \frac{dq'}{4\pi\epsilon_0 r}$$

Thus $\vec{E}$ is a perfect gradient, which we write as

$\vec{E} = -\nabla V$, where $V = \int_{R'} \frac{dq'}{4\pi\epsilon_0 r}$

$\underline{\mathrm{IV} \Rightarrow \mathrm{I}}$

Just like ($\mathrm{I} \to \mathrm{IV}$), using the definition $\vec{E} = -\nabla V$

$$\vec{E} = -\nabla \int_{R'} \frac{dq'}{4\pi\epsilon_0 r} = \int_{R'} \frac{dq'}{4\pi\epsilon_0} \nabla_r \frac{-1}{r} = \int_{R'} \frac{dq' \, \hat{r}}{4\pi\epsilon_0 r^2}$$

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III $\Leftrightarrow$ V

If $\vec{E} = -\nabla V$ then $\nabla \times \vec{E} = \nabla \times (-\nabla V) = 0$ always.

Conversely, by the inverse Poincaré lemma, (or whatever you want to call it)

$$\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$$

ie, $\nabla \times \vec{E} = 0$ implies that there is some scalar function $V(\vec{r})$ such that $\vec{E} = -\nabla V$.

Substitution of $\vec{E} = -\nabla V$ into the flux equation $\nabla \cdot E = \rho/\epsilon_0$ shows that

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \iff -\nabla \cdot \nabla V = \rho/\epsilon_0$$

IV $\Leftrightarrow$ V

By the Fundamental Theorem of Vector Calculus,

$$\int_a^b \nabla V \cdot d\vec{l} = V(\vec{b}) - V(\vec{a}) = \Delta V, \text{ and}$$

$$\int_a^b \vec{E} \cdot d\vec{l} \equiv \mathcal{E}_E \text{ by definition}$$

Since this is true for any path,

$$\mathcal{E} = -\Delta V \iff \vec{E} = -\nabla V$$

Taking the laplacian of the potential,

$$-\nabla^2 V = -\nabla^2 \int_{R'} \frac{dq'}{4\pi\varepsilon_0 \mathscr{r}} = \int_{R'} \frac{dq'}{\varepsilon_0} \nabla^2_{\vec{\mathscr{r}}} \frac{-1}{4\pi \mathscr{r}} = \int_{R'} \frac{dq'}{\varepsilon_0} \delta^3(\vec{\mathscr{r}})$$

$$= \frac{1}{\varepsilon_0} \int_{R'} d\tau' \rho(\vec{r}') \cdot \delta^3(\vec{r}-\vec{r}') = \rho(\vec{r})/\varepsilon_0$$

Thus $V = \int \frac{dq'}{4\pi\varepsilon_0 \mathscr{r}}$ implies that $-\nabla^2 V = \rho/\varepsilon_0$. or $\underline{\text{IV}} \Rightarrow \underline{\text{V}}$

Working backwards, to invert the equation $-\nabla^2 V = \rho/\varepsilon_0$ and solve for $V = -\nabla^{-2} \rho(\vec{r})/\varepsilon_0$, expand $\rho(\vec{r}) = \int_{R'} d\tau' \rho(\vec{r}') \delta(\vec{r}-\vec{r}')$ into a "forest" of delta functions (poles), and solve for the potential of each point charge individually:

$$-\nabla^{-2}\rho(\vec{r}) = -\nabla^{-2} \int_{R'} d\tau' \rho(\vec{r}') \delta(\vec{r}-\vec{r}')$$

$$= \int_{R'} d\tau' \rho(\vec{r}') \cdot -\nabla^{-2} \delta(\vec{\mathscr{r}})$$

$$= \int_{R'} d\tau' \rho(\vec{r}') \cdot \frac{1}{4\pi \mathscr{r}}. = \int_{R'} \frac{dq}{4\pi \mathscr{r}}$$

The point potential $\varepsilon_0 V_{pt} = \frac{1}{4\pi \mathscr{r}}$ is called a Green's function because it is the solution of

$$-\nabla^2 G(\mathscr{r}) = \delta^3(\vec{\mathscr{r}})$$

Thus we get $V = -\nabla^{-2}\rho/\varepsilon_0 = \int_{R'} \frac{dq'}{4\pi\varepsilon_0 \mathscr{r}}$, so $\underline{\text{V}} \Rightarrow \underline{\text{IV}}$.

$\underline{\text{III} \Rightarrow \text{I}}$ (This one uses just about everything else).

Assuming $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ and $\nabla \times \vec{E} = \vec{0}$, we can use the Helmholtz theorem to solve for $\vec{E}$ in terms of it divergence and curl.

$$\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E})$$

$$\vec{E} = -\nabla\underbrace{\left(-\nabla^{-2}\left(\overbrace{\nabla \cdot \vec{E}}^{\rho/\varepsilon_0}\right)\right)}_{V} + \nabla \times \left(-\nabla^{-2}\left(\cancel{\nabla \times \vec{E}}^{\,0}\right)\right) \quad \text{[Helmholtz]}$$

Thus we see that $\nabla \times \vec{E} = 0 \Rightarrow E = -\nabla V$,

where $V = -\nabla^{-2} \rho/\varepsilon_0$ (ie, $-\nabla^2 V = \rho/\varepsilon_0$)

We just solved this equation in ($\text{V} \Rightarrow \text{IV}$):

$$V = \int_{R'} \frac{dq'}{4\pi\varepsilon_0 \mathscr{r}}$$

Now just take the gradient ($\text{IV} \Rightarrow \text{I}$):

$$\vec{E} = -\nabla V = \int_{R'} \frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2},$$

which is Coulomb's law.

$\underline{\text{VI ?}}$

There is a sixth formulation, the differential form of Coulomb's law, formed by taking the gradient of Poisson's equation (V):

$$\nabla\left(-\nabla^2 V\right) = \nabla \rho/\varepsilon_0, \quad \text{or} \quad \boxed{\nabla^2 \vec{E} = \nabla \rho/\varepsilon_0}$$