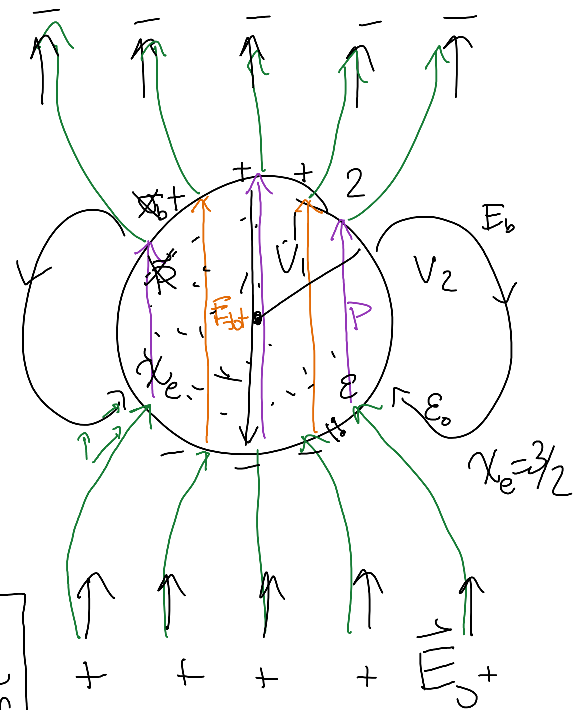
Example 4.7

Wednesday, March 19, 2014
10:33 AM

$V=\sum_{l=0}^{\infty}\left(A_l r^l + B r^{-l-1}\right) P_l(\cos\theta)$ sph ← Griffiths 4.7

$V=\sum_{m=1}^{\infty}\left(A_m s^m + B_m s^{-m}\right)\left(C_m \sin(m\phi)+D_m\cos(m\phi)\right)$ cyl ← HW7 #1



Bulk Differential Equations

$\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0(1+\chi_e)$

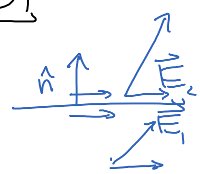
$\boxed{\begin{array}{l}\nabla\times\vec{E}=0\\ \nabla\cdot\vec{D}=\rho_f\end{array}} \Rightarrow \vec{E}=-\nabla V \quad \vec{D}=-\varepsilon\nabla V$

$-\nabla\cdot\varepsilon\nabla V=\rho_f=0$

$\vec{D}=\varepsilon\vec{E}$ $\boxed{\nabla^2 V=0}$

Boundary Conditions

$\nabla \to \Delta\hat{n}$ $\tau \to a$



$\Delta_{2-1}\,\hat{n}\times\vec{E}=\vec{0}$ $E_{2t}-E_{1t}=0$

$\Delta\hat{n}\cdot\vec{D}=\sigma_f$ $D^{\perp}_{2n}-D^{\perp}_{1n}=\sigma_f$

$\boxed{\begin{array}{l}V_1=V_2\\ -\varepsilon_2\frac{\partial V_2}{\partial n}+\varepsilon_1\frac{\partial V_1}{\partial n}=\sigma_f\end{array}}$

$\hat{n}\cdot\left(D=\varepsilon E=\varepsilon(-\nabla V)\right)=-\varepsilon\frac{\partial V}{\partial n}$

$\hat{n}\cdot D=(n\cdot\nabla)\cdots=\frac{\partial}{\partial n}$

General Solutions

$V_1=\sum_l\left(a_l r^l+\cancel{b_l r^{-l-1}}\right)P_l$ $V_2=\sum_l\left(c_l r^l+d_l r^{-l-1}\right)P_l$

Application of Boundary Conditions

i) $V(r\to 0)$ finite: $b_l=0$ for all l.

ii) $V(r\to\infty)$ blows up properly $\to$ $-E_0 z$ $\quad -\nabla V = E_0\hat{z}$

$V \to \underbrace{-E_0}\, r^1 P_1(\cos\theta)$ $\quad l=1$

$\Rightarrow$ $C_1 = -E_0$ $\quad C_0 = C_2 = C_3 = \dots = 0.$

$$V_1 = \sum_l a_l r^l P_l \qquad V_2 = -E_0 r P_1 + \sum_l d_l r^{-l-1} P_l$$

iii) $V_1(R) = V_2(R)$: $\sum_l a_l R^l \underbrace{P_l} = -E_0 R \underbrace{P_1} + \sum_l d_l R^{-l-1} \underbrace{P_l}$

$l \neq 1$: $\boxed{\underline{a_l} R^l = \underline{d_l} R^{-l-1}}$

$l=1$: $a_1 R = -E_0 R + d_1 R^{-2}$ $\qquad d_1/R^3 = a_1 + E_0$

iv) $-\varepsilon_2 \left.\frac{\partial V_2}{\partial r}\right|_R + \varepsilon_1 \left.\frac{\partial V_1}{\partial r}\right|_R = 0$

$$-\varepsilon_0 \Big(-E_0 P_1 + \sum_l (-l-1) d_l R^{-l-2} \underbrace{P_l}_{\text{basis}}\Big) + \varepsilon\Big(\sum_l (l) a_l R^{l-1} \underbrace{P_l}_{\text{basis.}}\Big) = 0$$

$l \neq 1$: $\boxed{-\varepsilon_0(-l-1) d_l R^{-l-2} + \varepsilon(l) a_l R^{l-1} = 0}$ $\Rightarrow a_l = d_l = 0.$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} a_l \\ d_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$l=1$: $\varepsilon_0 E_0 + 2\varepsilon_0 \underbrace{d_1/R^3} + \varepsilon a_1 = 0$ $\qquad d_1/R^3 = a_1 + E_0$

$\varepsilon_0 E_0 + 2\varepsilon_0(a_1 + E_0) + \varepsilon a_1 = 0$ $\qquad -(2\varepsilon_0 + \varepsilon)$

$a_1 = \dfrac{-3\varepsilon_0}{2\varepsilon_0+\varepsilon} E_0$ $\qquad d_1/R^3 = \left(\dfrac{-3\varepsilon_0}{2\varepsilon_0+\varepsilon} + \dfrac{2\varepsilon_0+\varepsilon}{2\varepsilon_0+\varepsilon}\right)E_0$

$\qquad = \left(\dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0$

Final Solution

$$V_1 = \frac{-3\varepsilon_0}{2\varepsilon_0+\varepsilon} z = \frac{-3}{3+\chi_e} E_0 r \cdot \cos\theta$$

$$= \underbrace{-E_0 z}_{\vec{E}_0} - \underbrace{\frac{\chi_e}{3+\chi_e} E_0 r\cos\theta}_{\vec{E}_b}$$

$$\text{so } \vec{P} = -3\varepsilon_0 E_b = \frac{3\chi_e}{3+\chi_e}\varepsilon_0 \vec{E}_0 = \chi_e \varepsilon_0 \vec{E}_{tot}$$

$$\vec{E}_{tot} = \frac{\vec{P}}{\chi_e} = \frac{3}{3+\chi_e}\vec{E}_0 \qquad \text{[inside]}$$

$$\vec{D} = \varepsilon_0 \vec{E}_{tot} + \vec{P} = \frac{3\varepsilon}{3+\chi_e}\vec{E}_0$$

$$V_2 = \underbrace{-E_0 z}_{} + \left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right) R^3 \boxed{\frac{\cos\theta}{r^2}} E_0$$

$$\vec{E}_2 = \vec{E}_0 + \frac{P}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

$$\vec{p} = \underbrace{\frac{4}{3}\pi R^3 \cdot 3\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\varepsilon_0 E_0}_{\frac{d\vec{p}}{d\tau} = \vec{P}}$$

$$\vec{P} = 3\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\varepsilon_0 \vec{E}_0$$

$$= 3\frac{\chi}{3+\chi}\varepsilon_0 \vec{E}_0$$