Exam 1

Wednesday, October 1, 2014
10:21 AM

[15 pts] 1. Short calculations.

a) Given a unit vector $\hat{\boldsymbol{n}} = [1, 1, 0]/\sqrt{2}$, project the vector $\vec{\boldsymbol{v}} = [4, 0, 0]$ into the sum of a vector parallel to $\hat{\boldsymbol{n}}$ and another vector perpendicular to $\hat{\boldsymbol{n}}$.

6

$$\vec{V}_{\parallel} = \hat{n}\,\hat{n}\cdot\vec{V} = \tfrac{1}{\sqrt{2}}(1,1,0)\ \tfrac{1}{\sqrt{2}}(1,1,0)\cdot(4,0,0) = 2(1,1,0) = (2,2,0)$$

$$\vec{V}_{\perp} = \vec{V} - \vec{V}_{\parallel} = [4,0,0] - [2,2,0] = [2,-2,0]$$

ALTERNATE SOLUTION:

$$\hat{n}\times\vec{V} = \tfrac{1}{\sqrt{2}}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 4 & 0 & 0 \end{vmatrix} = -\hat{z}\,\tfrac{4}{\sqrt{2}}$$

$$\hat{n}\times\hat{n}\times\vec{V} = \tfrac{1}{\sqrt{2}}\tfrac{4}{\sqrt{2}}\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2[1,-1,0] = [2,-2,0]$$

b) Transform the vector $\vec{\boldsymbol{F}}(x, y, z) = \hat{z}x$ completely into spherical coordinates.

5

$$\hat{z}\,x = (\hat{r}\,c_\theta - \hat{\theta}\,s_\theta)(r\,s_\theta c_\phi)$$

c) Simplify $\int_{-\infty}^{\infty} dx\, x^2\, \delta(x-2)$.

4

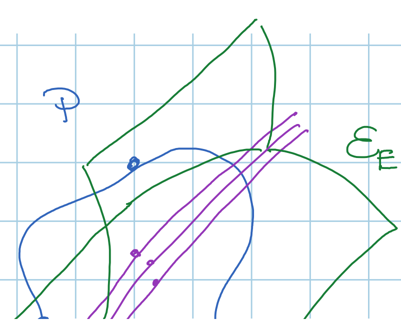
$$\int_{-\infty}^{\infty} dx\ x^2\,\delta(x-2) = \int_{-\infty}^{\infty} du\,(u+2)^2\,\delta(u) = 2^2 = 4$$

[20 pts] 2. Essay question: answer the following questions in paragraph form. You may illustrate your description, but will also be graded on your written response.

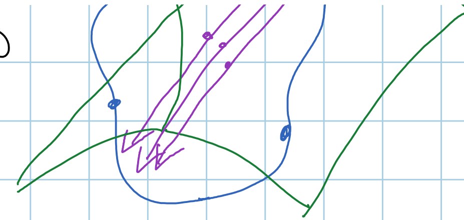
a) What are the geometric relations between a vector field $\vec{\boldsymbol{E}}(\vec{r})$, its flow surfaces, its curl $\boldsymbol{\nabla}\times\vec{\boldsymbol{E}}$, and line integral $\int_{\mathcal{P}} \vec{\boldsymbol{E}}\cdot d\vec{\boldsymbol{l}}$? Explain Stokes's theorem and the existence of a scalar potential in terms of this geometry.

2

The flow sheets of a vector field are surfaces perpendicular to the direction of the field spaced so that the "flow density" of the stacked sheets equals

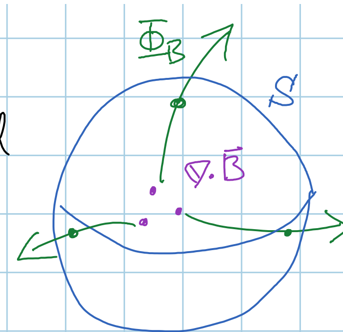


of the field spaced so that the "flow density" of the stacked sheets equals the magnitude at each field point.



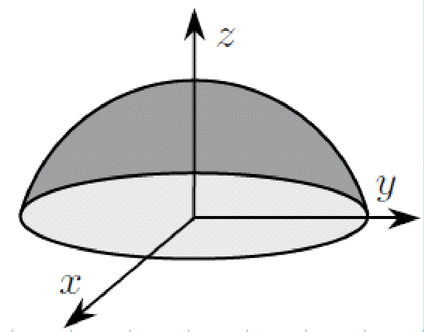
2 The curl of the field is a flux field represented by flux lines at the boundary of each flow surface. The line integral (flow) along
2 a path is the algebraic sum of surfaces crossed by the path. Stokes theorem states that the number
2 of curl flux lines through a closed loop equals the number of flow surfaces crossed by the path, since both are connected geometrically. If there is no curl, the flow sheets have no boundaries and represent equipotential surfaces

b) What are the geometric relations between an axial vector field $\vec{B}(\vec{r})$, its flux lines, its divergence $\nabla \cdot \vec{B}$, and surface integral $\int_S \vec{B} \cdot \vec{da}$? Explain Gauss' theorem and the existence of a vector potential in terms of this geometry.



The flux tubes of a field follow the
2 direction of the field and are spaced such that the "flux density" equals the magnitude of the field. Its
2 divergence equals the density of dots at the boundary of each field line. The surface integral
2 "flux" equals the number of field lines passing through the surface. Gauss' law states that the flux through
2 a closed surface equals the total divergence (dots) enclosed within the surface, again because they are connected.
2 If the field lines form closed loops, they form the boundary of flow sheets of a potential.

[20 pts] 3. Calculate the integral $\int_{\mathcal{S}} \nabla\times(\hat{r}\,\sin\theta + \hat{\phi}\,r)\cdot d\vec{a}$, where $\mathcal{S}$ is top half of a sphere of radius $r = 2$ about the origin.



5 $d\vec{a} = d\vec{l}_\theta \times d\vec{l}_\phi = \hat{\theta}\, r\, d\theta \times \hat{\phi}\, r s_\theta\, d\phi$

$= \hat{r}\, r^2 s_\theta\, d\theta\, d\phi$

5 $\nabla\times(\hat{r} s_\theta + \hat{\phi} r) = \dfrac{1}{r^2 s_\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r s_\theta \hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ s_\theta & 0 & r s_\theta r \end{vmatrix} = \dfrac{1}{r^2 s_\theta}\left[\hat{r}\, r^2 c_\theta - \hat{\theta}\, 2 r^2 s_\theta - \hat{\phi}\, r s_\theta c_\theta\right]$

5 $\int_S \nabla\times(\hat{r} s_\theta + \hat{\phi} r)\cdot d\vec{a} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \dfrac{\hat{r}\, r^2 c_\theta}{\cancel{r^2 s_\theta}} + \cdots \cdot \hat{r}\, \cancel{r^2 s_\theta}\, d\theta\, d\phi = 2\pi \int_{\theta=0}^{\pi/2} r^2 c_\theta\, d\theta = 8\pi$

b) Verify Stokes' theorem by calculating the corresponding line integral.

5 $\int_{\partial S} d\vec{l}\cdot(\hat{r} s_\theta + \hat{\phi} r) = \int_{\phi=0}^{2\pi} \hat{\phi}\, r s_\theta\, d\phi \cdot (\hat{r} s_\theta + \hat{\phi} r)$

$= \int_{\phi=0}^{2\pi} r^2 s_\theta\, d\phi = 2\pi r^2 s_\theta = 8\pi$

ALTERNATE SOLUTION:

on sphere

8 $\nabla\times\vec{V}\cdot d\vec{a} = d(\vec{V}\cdot d\vec{l}) = d(\cancel{s_\theta dr} + r^2 s_\theta\, d\phi) = r^2 c_\theta\, d\theta\, d\phi$

6 a) $\int_S \nabla\times\vec{V}\cdot d\vec{a} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} r^2 c_\theta\, d\theta\, d\phi = 2\pi r^2 = 8\pi$

6 b) $\int_{\partial S} \vec{V}\cdot d\vec{l} = \int_{\phi=0}^{2\pi} r^2 s_\theta\, d\phi = 2\pi r^2 = 8\pi$

[20 pts] 4. a) Show that the divergence of an inverse square force field has the singularity:

$$\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}).$$

4 $\nabla\cdot\dfrac{\hat{r}}{r^2} = \dfrac{1}{r^2 s_\theta}\left[\partial_r\left(\cancel{r^2} s_\theta \dfrac{1}{\cancel{r^2}}\right) + \partial_\theta \cdots + \partial_\phi \cdots\right] = 0$

4 $\int_V \nabla \cdot \frac{\hat{r}}{r^2}\, d\tau = \oint_{\partial V} d\vec{a} \cdot \frac{\hat{r}}{r^2} = \oint r^2 d\Omega \frac{1}{r^2} = \oint_{\text{sphere}} d\Omega = 4\pi$

thus $\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\, \delta^3(\vec{r})$

b) Using $-\nabla^2 \frac{1}{4\pi r} = \delta^3(\vec{\mathscr{r}}\,)$, and properties of $\delta^3(\vec{\mathscr{r}}\,)$ where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, show for any field $\rho(\vec{r})$ that

$$-\nabla^{-2}\rho(\vec{r}) = \int_{\vec{r}\,'} \frac{d\tau'\, \rho(\vec{r}\,')}{4\pi \mathscr{r}}$$

$$\rho(\vec{r}) = \int d\tau'\, \rho(\vec{r}\,')\, \delta^3(\vec{r} - \vec{r}\,')$$

6 $$-\nabla^{-2}\rho(\vec{r}) = \int d\tau'\, \rho(\vec{r}\,') \left[-\nabla^{-2} \delta^3(\vec{\mathscr{r}})\right] \qquad -\nabla^2 \frac{1}{4\pi\mathscr{r}} = \delta^3(\vec{\mathscr{r}})$$

$$= \int d\tau'\, \rho(\vec{r}\,') \frac{1}{4\pi\mathscr{r}} = \int \frac{d\tau'\, \rho(\vec{r}\,')}{4\pi\mathscr{r}}$$

c) Derive the Helmholtz theorem: that any vector field $\vec{F}(\vec{r})$ may be deomposed into longitudinal and transverse components expressed in terms of its gradient and curl: [circle the potentials below]

$$\vec{F} = -\nabla \underbrace{\int_{\vec{r}\,'} \frac{d\tau'\, \nabla' \cdot \vec{F}(\vec{r}\,')}{4\pi\mathscr{r}}}_{=V} + \nabla \times \underbrace{\int_{\vec{r}\,'} \frac{d\tau'\, \nabla' \times \vec{F}(\vec{r}\,')}{4\pi\mathscr{r}}}_{=\vec{A}}$$

$$\nabla^2 \vec{F} = \nabla \nabla \cdot \vec{F} - \nabla \times \nabla \times \vec{F}$$

6 $$\vec{F} = -\nabla\left(-\nabla^{-2}\, \nabla \cdot \vec{F}\right) + \nabla \times \left(-\nabla^{-2}\, \nabla \times \vec{F}\right)$$

$$= -\nabla \int \frac{d\tau'\, \nabla' \cdot \vec{F}'}{4\pi\mathscr{r}} + \nabla \times \int \frac{d\tau'\, \nabla' \times \vec{F}'}{4\pi\mathscr{r}} \qquad \text{from (b)}$$