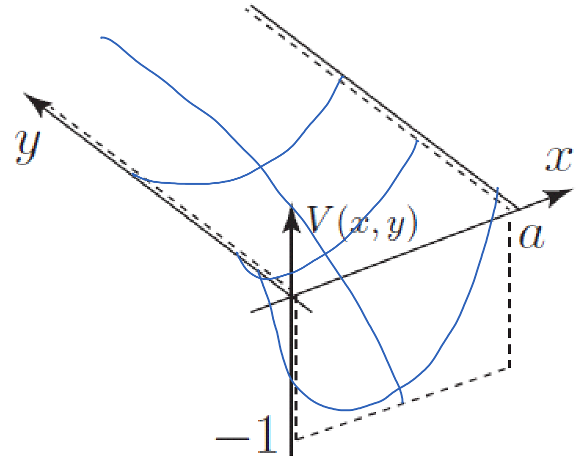
Exam 3

Wednesday, November 19, 2014
10:03 AM

[10 pts] 1. Calculate the first two non-zero terms of the general solution for the electric potential $V(x, y)$ in the region $0 < x < a$ and $0 < y < \infty$, with the boundary conditions $V(0, y) = V(a, y) = 0$ and $V(x, 0) = -1$. Plot the solution.



$$X(x) = A\cos(kx) + B\sin(kx)$$

$$X(0) = 0 \Rightarrow A = 0$$

$$Y(a) = 0 \Rightarrow \sin(ka) = 0 \Rightarrow k_n a = n\pi \quad n = 1, 2, 3\ldots$$

$$V(x,y) = \sum_n \left(A_n e^{k_n y} + B_n e^{-k_n y}\right)\sin(k_n x)$$

$$V(x,\infty) \text{ finite} \Rightarrow A_n = 0$$

$$V(x,0) = 1 \qquad \int_0^a \sin(k_m x) \sum_n B_n e^0 \sin(k_n x)\,dx = \int_0^a \sin k_m x \cdot -1\,dx$$

$$\sum_n B_n \delta_{mn} \frac{a}{2} = -\int_0^a \sin k_m x\,dx$$

$$B_m = \frac{-2}{a}\int_0^a \sin k_m x\,dx = \frac{+2}{a}\frac{1}{k_m}\cos k_m x\Big|_0^a$$

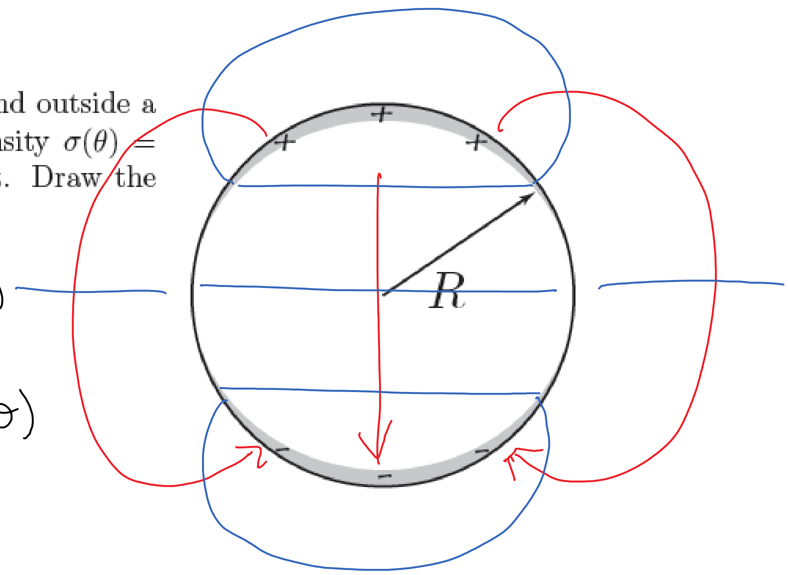
$$B_1 = \frac{2}{a}\frac{1}{k_m}\cdot(-2) \qquad B_2 = 0 \qquad B_3 = \frac{2}{a}\frac{1}{k_m}(-2)$$

$$= \frac{-4}{\pi} \qquad\qquad = \frac{-4}{3\pi}$$

[5 pts] 2. Describe the multipole expansion: what it is, how we get it, and why it is useful.

It is an "Taylor" expansion of the potential about the origin (close to the charge). It represents the key features of a charge distribution, as a few numbers, each with a characteristic field pattern. Usually only the lowest nonzero multipole is significant. The expansion separates source variables from field variables.

[15 pts] 3. Calculate the electric potential inside and outside a hollow sphere of radius R with surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$. Calculate the electric field inside and out. Draw the electric field lines and equipotentials everywhere.



$$V_1(r,\theta) = \sum_l \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)$$

$$V_2(r,\theta) = \sum_l \left(C_l r^l + D_l r^{-l-1}\right) P_l(\cos\theta)$$

i) $r \to 0$ V finite $\to B_l = 0$

ii) $r \to \infty$ V finite $\to C_l = 0$

iii) $V_1(R) - V_2(R) = 0$: $\sum_l \left(A_l R^l - D_l R^{-l-1}\right) P_l = 0$

$$D_l = A_l R^{2l+1}$$

iv) $-\frac{\partial V_2}{\partial R} + \frac{\partial V_1}{\partial R} = \sigma/\varepsilon_0$: $\sum_l \left[D_l(-l-1)R^{-l-2} + A_l(l)R^{l-1}\right] P_l = \sigma_0/\varepsilon_0 \cos\theta$

$$\sum_l (2l+1) R^{l-1} A_l P_l = \sigma_0/\varepsilon_0 \, P_1$$

$$3A_1 = \sigma_0/\varepsilon_0 \qquad A_l = \frac{\sigma_0}{3\varepsilon_0}\delta_{l1}$$

$$V_1(r,\theta) = \frac{\sigma_0}{3\varepsilon_0} r\cos\theta = \frac{\sigma_0 z}{3\varepsilon_0} \qquad \vec{E}_1 = -\nabla V_1 = -\frac{\sigma_0}{3\varepsilon_0}\hat{z}$$

$$V_2(r,\theta) = \frac{\sigma_0 R^3}{3\varepsilon_0}\frac{\cos\theta}{r^2} \qquad \vec{E}_2 = -\nabla V_2 = \frac{\sigma_0}{3\varepsilon_0}\left(\hat{r}\cdot 2\cos\theta + \hat{\theta}\sin\theta\right)$$

$$= \frac{\sigma_0 R^3}{3\varepsilon_0 r^3}\left(3\hat{r}\hat{r}\cdot\hat{z} - \hat{z}\right)$$

[10 pts] 4. Calculate the dipole moment of the above sphere and relate it the interior and exterior potential calculated above.

$$p_z = \int dq'\, r'\cos\theta' = \int \sigma_0 \cos\theta'\, R^2 \sin\theta' d\theta' d\phi' \cdot R\cos\theta' \qquad \text{let } x = \cos\theta'$$

$$= 2\pi\sigma_0 R^3 \cdot \int_{-1}^{1} x^2 dx = \frac{4\pi}{3} R^3 \sigma_0$$

$$V_1 = \frac{\sigma_0}{3\varepsilon_0} z = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 R^3} \qquad V_2 = \frac{\sigma_0 R^3}{3\varepsilon_0}\frac{\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$