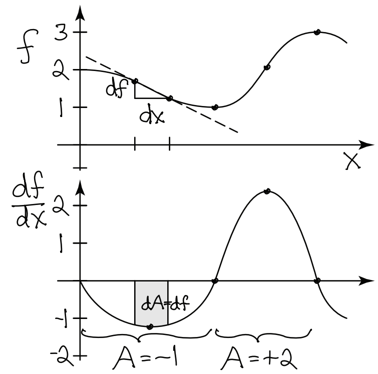
Section 1.2 - Differential Calculus



* differential operator

~ ex. $u = x^2 \quad du = dx^2 = 2x\,dx$ $\qquad d \equiv \lim_{\Delta\to 0} \Delta \approx 0$

or $d(\sin x^2) = \cos(x^2)\,d\,x^2 = \cos x^2 \cdot 2x \cdot dx$

~ df and dx connected - refer to the same two endpoints

~ made finite by taking ratios (derivative or chain rule)

or inifinite sum = integral (Fundamental Thereom of calculus)

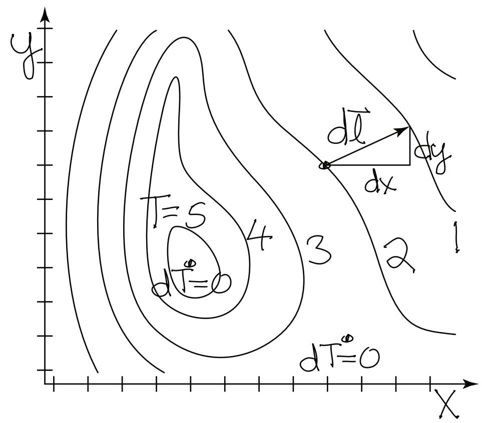
$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \int_a^b \frac{df}{dx}\,dx = \int_a^b df = f\Big|_a^b$$

* scalar and vector fields - functions of position ($\vec{r}$)

~ "field of corn" has a corn stalk at each point in the field

~ scalar fields represented by level curves (2d) or surfaces (3d)

~ vector fields represented by arrows, field lines, or equipotentials



* partial derivative & chain rule

~ signifies one varying variable AND other fixed variables

~ notation determined by denominator; numerator along for the ride

~ total variation split into sum of variations in each direction

$$\frac{\partial u}{\partial x} \quad \left(\frac{\partial u}{\partial x}\right)_{y,z} \quad \partial_x u \quad u_{,x} \qquad\qquad \frac{\cdots}{\cdots} = \frac{dx}{\cdots}\frac{\cdots}{\partial x} + \frac{dy}{\cdots}\frac{\cdots}{\partial y} + \frac{dz}{\cdots}\frac{\cdots}{\partial z}$$

* vector differential - gradient

~ differential operator , del operator

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz$$

$$= \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla}\,T \cdot \underbrace{(dx, dy, dz)}_{\vec{d\ell}}$$

$$\boxed{\begin{aligned} d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} &&= d\vec{r}\cdot\nabla \\ \nabla &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} &&= \frac{d}{d\vec{r}} \\ \vec{d\ell} &= \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz &&= d\vec{r} \end{aligned}}$$

~ differential line element: $\cdot\vec{d\ell}$ and $\frac{\;}{\vec{d\ell}}$ transforms between $\hat{x},\hat{y},\hat{z} \leftrightarrow dx, dy, dz$ and $d \leftrightarrow \nabla$

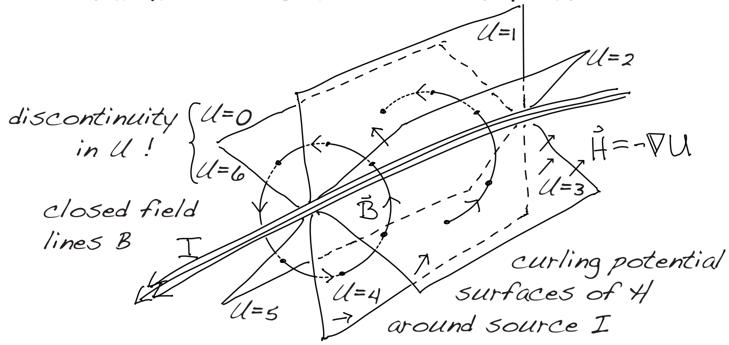
~ example: $d\,x^2y = 2xy\,dx + x^2\,dy = (2xy,\ x^2)\cdot(dx, dy)$

~ example: let $z = f(x,y)$ be the graph of a surface. What direction does ∇f point?

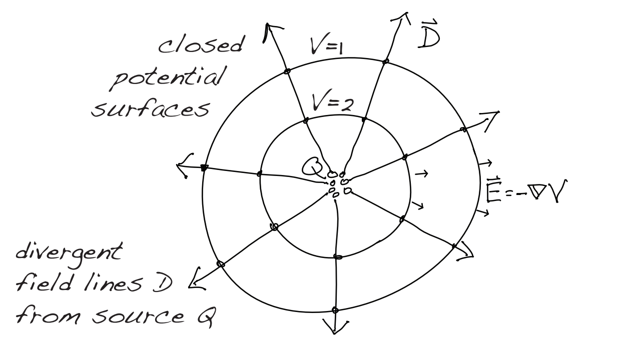
now let $g = z - f(x,y)$ so that $g = 0$ on the surface of the graph

then $\nabla g = (-f_{,x}, -f_{,y}, 1)$ is normal to the surface

* illustration of curl - flow sheets



* illustration of divergence - flux tubes



Higher Dimensional Derivatives

* curl – circular flow of a vector field

$$\nabla\times\vec V = \begin{vmatrix}\hat x & \hat y & \hat z\\ \partial_x & \partial_y & \partial_z\\ V_x & V_y & V_z\end{vmatrix} = \begin{matrix}\hat x\,(V_{z,y}-V_{y,z})\\ +\hat y\,(V_{x,z}-V_{z,x})\\ +\hat z\,(V_{y,x}-V_{x,y})\end{matrix}$$

* divergence – radial flow of a vector field

$$\nabla\cdot\vec V = (\partial_x\ \partial_y\ \partial_z)\begin{pmatrix}V_x\\ V_y\\ V_z\end{pmatrix} = V_{x,x}+V_{y,y}+V_{z,z}$$

* product rules

~ how many are there?

~ examples of proofs

$$\vec a\times(\vec b\times\vec c) = \vec b(\vec a\cdot\vec c) - \vec c(\vec a\cdot\vec b)$$

$$\vec A\times(\nabla\times\vec B) = \nabla(\vec A\cdot\vec B) - \vec B(\vec A\cdot\nabla)$$

$$\nabla\times(\vec A\times B) = \vec A(\nabla\cdot\vec B) - \vec B(\nabla\cdot\vec A)$$

$$\nabla(fg) = \nabla f\cdot g + f\cdot\nabla g$$

$$\nabla(\vec A\cdot\vec B) = \vec A\times(\nabla\times\vec B) + (\vec A\cdot\nabla)\vec B + (\vec B\leftrightarrow\vec A)$$

$$\nabla\times(f\vec A) = \nabla f\times\vec A + f(\nabla\times\vec A)$$

$$\nabla\times(\vec A\times\vec B) = (B\cdot\nabla)A - B(\nabla\cdot A) - (\vec B\leftrightarrow\vec A)$$

$$\nabla\cdot(f\vec A) = \nabla f\cdot\vec A + f\,\nabla\cdot\vec A$$

$$\nabla\cdot(\vec A\times\vec B) = (\nabla\times\vec A)\cdot\vec B - \vec A\cdot(\nabla\times\vec B)$$

* second derivatives – there is really only ONE! (the Laplacian) $\boxed{\nabla^2\equiv\nabla\cdot\nabla\equiv\partial_x^2+\partial_y^2+\partial_z^2}$

1) $\nabla\cdot(\nabla T) = \nabla^2 T$ ~ eg: $\nabla^2 T = 0$ no net curvature – stretched elastic band

$(\nabla\cdot\nabla)\vec v = \nabla^2\vec v$ ~ defined component-wise on v_x, v_y, v_z (only cartesian coords)

3), 5) $\nabla^2 = \nabla_\parallel^2 + \nabla_\perp^2$ ~ longitudinal / transverse projections $\nabla(\nabla\cdot\vec v)\equiv\nabla_\parallel^2\vec v$

$= \nabla(\nabla\cdot \; -\nabla\times\nabla\times$ $\quad\vec K\cdot\vec K = \vec K\vec K\cdot\; - \vec K\times(\vec K\times$ $\quad -\nabla\times\nabla\times\vec v\equiv -\nabla_\perp^2\vec v$

2), 4) $\nabla\times\nabla = 0$ $\quad\nabla\times(\nabla T) = 0$ $\quad\nabla\cdot(\nabla\times\vec v) = 0$

$$\nabla\times\nabla = \begin{vmatrix}\hat x & \hat y & \hat z\\ \partial_x & \partial_y & \partial_z\\ \partial_x & \partial_y & \partial_z\end{vmatrix} = \begin{matrix}\hat x\,(\partial_y\partial_z-\partial_z\partial_y)\\ +\hat y\,(\partial_z\partial_x-\partial_x\partial_z)\\ +\hat z\,(\partial_x\partial_y-\partial_y\partial_x)\end{matrix}$$

~ equality of mixed partials ($d^2=0$)

* unified approach to all higher-order derivatives with differential operator

1) $d^2 = 0$ 2) $dx^2 = 0$ 3) $dx\,dy = -dy\,dx$ + differential (line, area, volume) elements

~ Gradient

$$df = f_{,x}dx + f_{,y}dy + f_{,z}dz = \nabla f\cdot d\vec l \qquad d\vec l = (dx,dy,dz) = d\vec r$$

~ Curl

$$\begin{aligned}d(\vec A\cdot d\vec l) &= d(A_x dx + A_y dy + A_z dz)\\ &= A_{x,x}\,\cancel{dx}dx + A_{x,y}\,dy\,dx + A_{x,z}\,dz\,dx\\ &\quad + A_{y,x}\,dx\,dy + A_{y,y}\,\cancel{dy}dy + A_{y,z}\,dz\,dy\\ &\quad + A_{z,x}\,dx\,dz + A_{z,y}\,dy\,dz + A_{z,z}\,\cancel{dz}dz\\ &= (A_{z,y}-A_{y,z})dy\,dz + (A_{x,z}-A_{z,x})dz\,dx + (A_{y,x}-A_{x,y})dx\,dy\end{aligned}$$

$$d(\vec A\cdot d\vec l) = (\nabla\times\vec A)\cdot d\vec a$$

$$d\vec a = (dy\,dz,\ dz\,dx,\ dx\,dy) = \tfrac12 d\vec l\times d\vec l = d^2\vec r$$

~ Divergence

$$\begin{aligned}d(\vec B\cdot d\vec a) &= d(B_x dy\,dz + B_y dz\,dx + B_z dx\,dy)\\ &= B_{x,x}\,dx\,dy\,dz + B_{x,y}\,dy\,\cancel{dy}\,dz + B_{x,z}\,dz\,dy\,\cancel{dz}\\ &\quad + B_{y,x}\,dx\,\cancel{dz}\,dx + B_{y,y}\,dy\,dz\,dx + B_{y,z}\,dz\,\cancel{dz}\,dx\\ &\quad + B_{z,x}\,dx\,\cancel{dx}\,dy + B_{z,y}\,dy\,\cancel{dx}\,dy + B_{z,z}\,dz\,dx\,dy\\ &= (B_{x,x}+B_{y,y}+B_{z,z})\,dx\,dy\,dz\end{aligned}$$

$$d(\vec B\cdot d\vec a) = \nabla\cdot\vec B\,d\tau \qquad d\tau = \tfrac16 d\vec l\cdot d\vec l\times d\vec l = d^3\vec r$$

$$\boxed{\nabla f = \frac{df}{d\vec l} = \frac{df}{d\vec r}} \qquad \boxed{\nabla\times\vec A = \frac{d(\vec A\cdot d\vec l)}{d\vec a} = \frac{d(d\vec r\cdot\vec A)}{d^2\vec r}} \qquad \boxed{\nabla\cdot\vec B = \frac{d(\vec B\cdot d\vec a)}{d\tau} = \frac{d(d^2\vec r\cdot\vec B)}{d^3\vec r}}$$