Section 1.3 - Integration

* different types of integration in vector calculus

1-dim: $\omega^{(1)} = \lambda\, dl,\ \varphi\, d\vec{l},\ \vec{A}\, dl,\ \vec{A}\cdot d\vec{l},\ \vec{A}\times d\vec{l}$

2-dim: $\omega^{(2)} = \sigma\, da,\ \sigma\, d\vec{a},\ \vec{B}\, da\ \ \vec{B}\cdot d\vec{a},\ \vec{B}\times d\vec{a}$

3-dim: $\omega^{(3)} = \rho\, d\tau,\quad \vec{F}\, d\tau$

Flow: $\mathcal{E}_A = \int \tilde{A} = \int \vec{A}\cdot d\vec{l}$

Flux: $\Phi_B = \int \tilde{B} = \int \vec{B}\cdot d\vec{a}$

Substance: $Q_\rho = \int \tilde{\rho} = \int \rho\, d\tau$

~ "differential forms" are everything after the '$\int$'
all have a 'd' somewhere inside

~ often $d\vec{l}, d\vec{a}, d\tau$ are burried inside of another 'd'

current element $dq \equiv q_i^{(0)}, \lambda\, dl^{(1)}, \sigma\, da^{(2)}, \rho\, d\tau^{(3)}$

charge element $d\vec{q} \equiv \vec{v}q_i, I\, d\vec{l}, \vec{K}\, da, \vec{J}\, d\tau$

$d\vec{l}_{rec} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$d\vec{a}_{rec} = \hat{x}\,dy\,dz + \hat{y}\,dz\,dx + \hat{z}\,dx\,dy$

$d\tau_{rec} = dx\,dy\,dz$

~ two types of regions:

over the region R: $\int_R \omega$ (open region)

over the boundary ∂R of R: $\oint_{\partial R} \omega$ (closed region)

* recipe for ALL types of integration

a) Parametrize the region

~ parametric vs relations equations of a region

~ boundaries translate to endpoints on integrals

coordinates on path/surface/volume

1-d $\mathcal{P}$: $\vec{r}(t)$

2-d $\mathcal{S}$ $\vec{r}(s,t)$

3-d $\mathcal{V}$ $\vec{r}(s,t,u)$

boundary of coordinates

$\int_{s=a}^{b}\int_{t=t_0(s)}^{t_1(s)}$

b) Pull back the paramters

~ x,y,z become functions of s,t,u

~ differentials: dx,dy,dz become ds,dt,du

~ reduce using the chain rule

$d\vec{l} = \frac{d\vec{r}}{dt}dt$

$d\vec{a} = \frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\,ds\,dt$

$d\tau = \frac{\partial\vec{r}}{\partial s}\cdot\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u}\,ds\,dt\,du$

$x = x(t) \quad dx = x'\,dt$

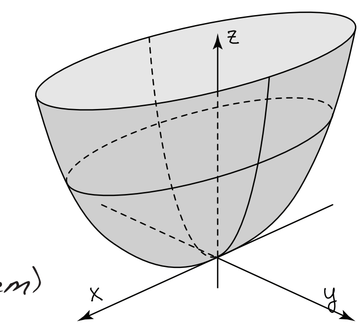
$y = y(t) \quad dy = y'\,dt$

$z = z(t) \quad dz = z'\,dt$

$$\int_R \vec{A}\cdot d\vec{l} = \int_{\vec{x}(t)} A_x(x,y,z)\,dx + A_y(x,y,z)\,dy + A_z(x,y,z)\,dz$$

$$= \int_{t=a}^{b} A_x(x(t),y(t),z(t))\frac{dx}{dt}dt + A_y(x(t),y(t),z(t))\frac{dy}{dt}dt$$

c) Integrate 1-d integrals using calculus of one variable



* example: line & surface integrals on a paraboloid (Stoke's theorem)

$\vec{A} = yz\hat{x}$ $\quad S:\ z = \frac{1}{4}x^2 + y^2 = s^2(c_\phi^2 + s_\phi^2)$

$0<z<1 \quad \partial S:\ 1 = \frac{1}{4}x^2 + y^2$

$x = 2s c_\phi \quad dx = 2ds\, c_\phi \ + -2s s_\phi\, d\phi$

$y = s\cdot s_\phi \quad dy = ds\, s_\phi \ + \ s\, c_\phi\, d\phi$

$z = s^2 \quad dz = 2s\cdot ds$

'partials'

$$d\vec{l} = \underbrace{\frac{\partial\vec{r}}{\partial s}\,ds} + \underbrace{\frac{\partial\vec{r}}{\partial\phi}\,d\phi} = d\vec{l}_s + d\vec{l}_\phi$$

$$d\vec{a} = d\vec{l}_s \times d\vec{l}_\phi = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 2c_\phi & s_\phi & 2s\\ -2ss_\phi & sc_\phi & 0\end{vmatrix} ds\,d\phi$$

$$= \left(-\hat{x}\,2s^2c_\phi - \hat{y}\,4s^2s_\phi + \hat{z}\,2s\right)ds\,d\phi$$

$\partial S:\ \vec{r}(s,\phi) \quad s=1 \quad ds=0 \quad d\vec{l} = d\vec{l}_\phi\ (s=1)$

$$\oint_{\partial S}\vec{A}\cdot d\vec{l} = \int_{\partial S} yz\,dx = -2\int_0^{2\pi}\overbrace{s_\phi^2}^{1/2}\,d\phi = -2\pi$$

$$\int_S \nabla\times\vec{A}\cdot d\vec{a} = \int_S(\hat{y}\partial_z - \hat{z}\partial_y)yz\cdot d\vec{a} = \int_S y\,da_y - z\,da_z$$

$$= \int_0^1\int_0^{2\pi}\left(s\cdot s_\phi \cdot -4s^2s_\phi - s^2\cdot 2s\right)ds\,d\phi$$

$$= \int_0^1 ds\int_0^{2\pi} -4s^3\underbrace{s_\phi^2}_{1/2} - 2s^3\,d\phi$$

$$= \int_0^1 -4s^3\cdot ds\cdot 2\pi = \frac{-4s^4}{4}\Big|_0^1\cdot 2\pi = -2\pi$$

* alternate method: substitute for dx, dy, dz (antisymmetric)

$$\int_S y\,dz\,dx - z\,dx\,dy = \int_S s s_\phi\cdot 2s\,ds\cdot(2c_\phi ds - 2ss_\phi d\phi) - s^2(2c_\phi ds - 2ss_\phi d\phi)(s_\phi ds + sc_\phi d\phi)$$

$$= \int_S -4s^3s_\phi^2\,ds\,d\phi - 2s^3c_\phi^2\,ds\,d\phi + 2s^3s_\phi^2\,\underbrace{d\phi\,ds}_{-ds\,d\phi}$$

$$= \int_S(-6s_\phi^2 - 2c_\phi^2)\,s^3\,ds\,d\phi$$

Flux, Flow, and Substance

* Differential forms

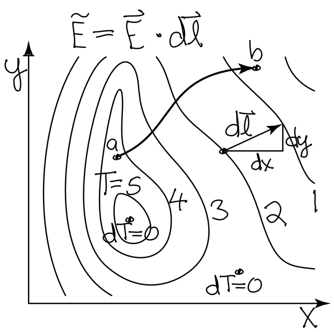
		Name	Geometrical picture
scalar:	$\varphi = \varphi(x)$		level curves
vector:	$d\mathcal{E} = \vec{A}\cdot d\vec{l} = A_x\,dx + A_y\,dy + A_z\,dz$		equipotentials (flow sheets)
pseudovector:	$d\Phi = \vec{B}\cdot d\vec{a} = B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy$		fieldlines (flux tubes)
pseudoscalar:	$dq = \rho\, d\tau = \rho\, dx\, dy\, dz$		boxes of substance

* Derivative 'd'

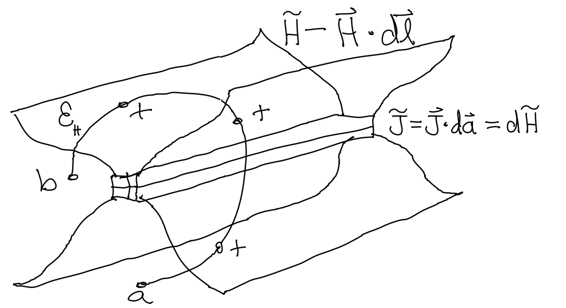
scalar:	$d\,\varphi = \nabla\varphi\cdot d\vec{l}$	grad	same equipotential surfaces
vector:	$d\,\vec{A}\cdot d\vec{l} = \nabla\times\vec{A}\cdot d\vec{a}$	curl	flux tubes at end of sheets
pseudovector:	$d\,\vec{B}\cdot d\vec{a} = \nabla\cdot\vec{B}\,d\tau$	div	boxes at the end of flux tubes
pseudoscalar:	$d\,\rho\, d\tau = 0$		

* Definite integral

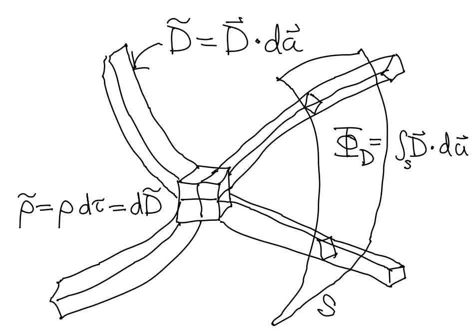
scalar:			
vector:	$\mathcal{E} = \int_P d\mathcal{E} = \int_P \vec{A}\cdot d\vec{l}$	flow	# of surfaces pierced by path
pseudovector:	$\Phi = \int_S d\Phi = \int_S \vec{B}\cdot d\vec{a}$	flux	# of tubes piercing surface
pseudoscalar:	$Q = \int_V dq = \int_V \rho\, d\tau$	subst	# of boxes inside volume



$$\Delta f = \int_a^b df = f\Big|_a^b = -4$$
$$\oint df = \Delta f = 0$$
$$df = \nabla f\cdot d\vec{l}$$



$$\mathcal{E}_H = \int_a^b \tilde{H} = \int_a^b \vec{H}\cdot d\vec{l} = +3$$
$$\mathcal{E}_H = \oint_{\partial R} \tilde{H} = \int_R d\tilde{H} = \int \tilde{J} = I = +4$$
$$d\tilde{H} = d(\vec{H}\cdot d\vec{l}) = (\nabla\times\vec{H})\cdot d\vec{a} = \vec{J}\cdot d\vec{a} = \tilde{J}$$



$$\Phi_D = \int_S \vec{D}\cdot d\vec{a} = \int_S \tilde{D} = +2.$$
$$\Phi_D = \oint_{\partial R} \tilde{D} = \int_R d\tilde{D} = \int_R \tilde{\rho} = Q = +4$$
$$d\tilde{D} = d(\vec{D}\cdot d\vec{a}) = \nabla\cdot\vec{D}\, d\tau = \rho\, d\tau = \tilde{\rho}$$

* Stoke's theorem

\# of flux tubes puncturing disk (S) bounded by closed path
EQUALS # of surfaces pierced by closed path (∂S)
~ each surface ends at its SOURCE flux tube

* Divergence theorem

\# of substance boxes found in volume (R) bounded by closed surface
EQUALS # of flux tubes piercin the closed surface (∂R)
~ each flux tube ends at its SOURCE substance box