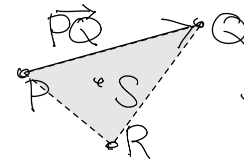
Section 1.4 - Affine Spaces

* Affine Space - linear space of points

POINTS vs VECTORS

~ operations $Q - P = \vec{V}$, $P + \vec{V} = Q$ $\qquad \vec{W} = \alpha\vec{u} + \beta\vec{v}$



$S = \alpha P + \beta Q + \gamma R$

$\alpha + \beta + \gamma = 1$

~ points are invariant under translation of the origin

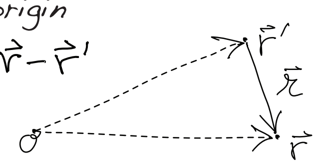
~ can treat points as vectors from the origin to the point (calculational trick)

cumbersome picture: many meaningless arrows from meaningless origin

position vector: field point $\vec{r} = (x, y, z)$ source pt $\vec{r}\,' = (x', y', z')$

displacement vector: $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$

differential: $d\vec{l} = \frac{\partial \vec{r}}{\partial q} dq = \vec{b}\, dq$



~ the only operation on points is the weighted average

weight w=0 for vectors and w=1 for points

~ transformation: affine vs linear

$$\begin{pmatrix} R & \vec{t} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{r} \\ 1 \end{pmatrix} = \begin{pmatrix} R\vec{r} + \vec{t} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} R & \vec{t} \\ 000 & 1 \end{pmatrix}\begin{pmatrix} \vec{v} \\ 0 \end{pmatrix} = \begin{pmatrix} R\vec{v} \\ 0 \end{pmatrix}$$

~ decomposition: coordinates vs components

- they appear the same for cartesian systems!
- coordinates are scalar fields $q^i(\vec{r})$

* Rectangular, Cylindrical and Spherical coordinate transformations

~ math: 2-d -> N-d physics: 3d + azimuthal symmetry

~ singularities on z-axis and origin

$S_\theta \equiv \sin\theta$

$C_\theta \equiv \cos\theta$

$$(\hat{s}, \hat{\phi}, \hat{z}) = (\hat{x}, \hat{y}, \hat{z}) \overbrace{\begin{pmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{R_{\hat{z}}(\phi)}$$

$x = s \cdot C_\phi$
$y = s \cdot S_\phi$

$s = r \cdot S_\theta$
$z = r \cdot C_\theta$

rect.		cyl.		sph.
x	$=$	$s \cdot C_\phi$	$=$	$r \cdot S_\theta \cdot C_\phi$
y	$=$	$s \cdot S_\phi$	$=$	$r \cdot S_\theta \cdot S_\phi$
z	$=$	z	$=$	$r \cdot C_\theta$

$$(\hat{r}\,\hat{\theta}\,\hat{\phi}) = (\hat{s}\,\hat{\phi}\,\hat{z}) \overbrace{\begin{pmatrix} S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \\ C_\theta & -S_\theta & 0 \end{pmatrix}}^{R_{\hat{\phi}}(\theta)} = (\hat{x}\,\hat{y}\,\hat{z})\, \overbrace{R_{\hat{z}}(\phi) \cdot R_{\hat{\phi}}(\theta)}^{R(\theta,\phi)}$$

$$R_{\hat{z}}(\phi) \cdot R_{\hat{\phi}}(\theta) = \begin{pmatrix} S_\theta C_\phi & C_\theta C_\phi & -S_\phi \\ S_\theta S_\phi & C_\theta S_\phi & C_\phi \\ C_\theta & -S_\theta & 0 \end{pmatrix}$$

$d\vec{l}_{rec} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$d\vec{l}_{cyl} = \hat{s}\,ds + \hat{\phi}\,s\,d\phi + z\,dz$

$d\vec{l}_{sph} = \hat{r}\underbrace{dr}_{dl_r} + \hat{\theta}\underbrace{r\,d\theta}_{dl_\theta} + \hat{\phi}\underbrace{r\sin\theta\,d\phi}_{dl_\phi}$

$d\vec{a}_{rec} = \hat{x}\,dy\,dz + \hat{y}\,dz\,dx + \hat{z}\,dx\,dy$

$d\vec{a}_{cyl} = \hat{s}\,s\,d\phi\,dz + \hat{\phi}\,dz\,ds + \hat{z}\,ds\,s\,d\phi$

$d\vec{a}_{sph} = \hat{r}\,r\,d\theta\,r\sin\theta\,d\phi + \hat{\theta}\,r\sin\theta\,d\phi\,dr + \hat{\phi}\,dr\,r\,d\theta$

$d\tau_{rec} = dx\,dy\,dz$

$d\tau_{cyl} = ds \cdot s\,d\phi \cdot dz$

$d\tau_{sph} = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$
$= r^2\,dr\,d\Omega$