Curvilinear Coordinates

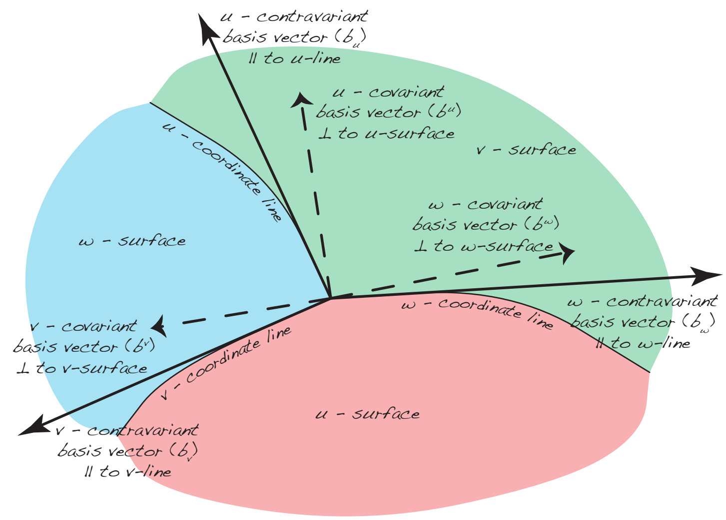
* coordinate surfaces and lines

~ each coordinate is a scalar field $q(\vec r)$

~ coordinate surfaces: constant q^i

~ coordinate lines: constant q^j, q^k

* coordinate basis vectors



$q^i \sim \{u, v, w\}$ ~ generalized coordinates

$\vec b_i = \left(\frac{\partial \vec r}{\partial q^i}\right)_{q^j, q^k} \sim \{\hat u f, \hat v g, \hat w h\}$ ~ contravariant basis

$\vec b^i = \nabla q^i \sim \{\hat u/f, \hat v/g, \hat w/h\}$ ~ covariant basis

$h_i = |\vec b_i| \sim \{f, g, h\}$ ~ scale factor

$\hat e_i = \vec b_i / h_i \sim \{\hat u, \hat v, \hat w\}$ ~ unit vector

$g_{ij} = \vec b_i \cdot \vec b_j \sim \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$ ~ metric (dot product)

$\vec\Gamma_{ij} = \frac{\partial \vec b_j}{\partial q^i} = \vec b_k \Gamma^k_{ij}$ ~ Christoffel symbols - derivative of basis vectors

* differential elements (orthogonal coords)

$$d\vec l = \frac{\partial \vec r}{\partial q^1} dq^1 + \frac{\partial \vec r}{\partial q^2} dq^2 + \frac{\partial \vec r}{\partial q^3} dq^3 = \vec b_i dq^i$$

$$= \hat e_1 \underbrace{h_1 dq^1}_{dl_1} + \hat e_2 \underbrace{h_2 dq^2}_{dl_2} + \hat e_3 \underbrace{h_3 dq^3}_{dl_3}$$

$$d\vec a = \tfrac12 d\vec l \times d\vec l = \begin{vmatrix} \hat e_1 & \hat e_2 & \hat e_3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \end{vmatrix}$$

$$= \hat e_1 \cdot h_2 dq^2 \cdot h_3 dq^3 + \hat e_2 \cdot h_3 dq^3 \cdot h_1 dq^1 + \hat e_3 \cdot h_1 dq^1 \cdot h_2 dq^2$$

$$d\tau = \tfrac12 d\vec l \times d\vec a = \tfrac12 d\vec l \cdot d\vec l \times d\vec l = h_1 dq^1 \cdot h_2 dq^2 \cdot h_3 dq^3$$

* example

$(c_\phi \equiv \cos\phi)$

$x = s\, c_\phi \qquad dx = c_\phi ds - s\, s_\phi d\phi$

$y = s\, s_\phi \qquad dy = s_\phi ds + s\, c_\phi d\phi$

$$d\vec l = \hat x dx + \hat y dy = (\hat x c_\phi + \hat y s_\phi) ds + (\hat x s_\phi - \hat y c_\phi) s\, d\phi$$

$$= \hat s\, ds + \hat\phi\, s\, d\phi \qquad (\hat s \;\; \hat\phi) = (\hat x \;\; \hat y) \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$$

$s^2 = x^2 + y^2 \qquad 2s\,ds = 2x\,dx + 2y\,dy$

$y = x \tan\phi \qquad dy = dx \tan\phi + x \sec^2\phi\, d\phi$

$$d\phi = \frac{-y}{s^2} dx + \frac{x}{s^2} dy$$

$$\nabla s = \frac{x}{s}\hat x + \frac{y}{s}\hat y = c_\phi \hat x + s_\phi \hat y = \hat s$$

$$\nabla\phi = \frac{-y}{s^2}\hat x + \frac{x}{s^2}\hat y = \frac{-s_\phi \hat x + c_\phi \hat y}{s} = \frac{\hat\phi}{s}$$

* formulas for vector derivatives in orthogonal curvilinear coordinates

$$df = \frac{\partial f}{\partial q^i} dq^i = \frac{\partial f}{h_i \partial q^i} \cdot h_i dq^i = \nabla f \cdot d\vec l$$

$$\nabla f = \frac{df}{d\vec r} = \frac{\hat e_i}{h_i} \frac{\partial}{\partial q^i} f$$

$$d(\vec A \cdot d\vec l) = d(A_k h_k dq^k) = \frac{\partial}{\partial q^j}(h_k A_k)\, dq^j dq^k$$

$$= \varepsilon_{ijk} \frac{\partial (h_k A_k)}{h_j h_k \partial q^k} da_i = (\nabla \times \vec A) \cdot d\vec a$$

$$\nabla \times \vec A = \frac{d(d\vec r \cdot \vec A)}{d^2 \vec r} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat e_1 & h_2 \hat e_2 & h_3 \hat e_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 A^1 & h_2 A^2 & h_3 A^3 \end{vmatrix}$$

$$d(\vec B \cdot d\vec a) = d(B_i h_j dq^j h_k dq^k) = \frac{\partial}{\partial q^i}(h_j h_k B_i)\, dq^i dq^j dq^k$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q} \frac{\partial (h_j h_k B_i)}{\partial q^i} d\tau = \nabla \cdot \vec B\, d\tau$$

$$\nabla \cdot \vec B = \frac{d(d^2\vec r \cdot \vec B)}{d^3 \vec r} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i}(h_j h_k B_i)$$

i, j, k cyclic

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q_i} f$$

this formula does not work for $\nabla^2 \vec B$

instead, use: $\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times$