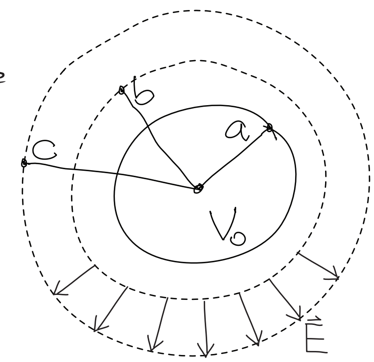
Section 3.2 - Method of Images

* concept: in a region $\mathcal{R}$, $V(\vec{r})$ depends ONLY on the boundary of V at $\partial\mathcal{R}$
  - ~ it doesn't matter how it was created, or where charge is outside $\mathcal{R}$
  - ~ more than one charge distribution can generate the same $V(\vec{r})$ inside $\mathcal{R}$

* Example 1: $V=V_0$ inside a constant sphere of radius a

  $V=\frac{q}{4\pi\varepsilon_0 r}$ for a point charge at the origin, OR on the outside of a uniform spherical shell of total charge q



  i) $q_0 = 4\pi\varepsilon_0 a \cdot V_0$ on a sphere of radius 'a' is one solution

  ii) $q_1 = 4\pi\varepsilon_0 b\, V_0$ on a sphere of radius 'b' another solution

  iii) how about $+q_2$ at radius 'b' and $-q_2$ at radius 'c'?

  for $r>c$, $V=\frac{+q_2}{4\pi\varepsilon_0 r}+\frac{-q_2}{4\pi\varepsilon_0 r}=0$ $\quad E=0$ for $r>c$

  and for $r<b$ $V=\frac{q_2}{4\pi\varepsilon_0 b}-\frac{q_2}{4\pi\varepsilon_0 c}=V_0=\frac{q_0}{4\pi\varepsilon_0 a}$, the equivalent charge of (i)

  so $q_2=\frac{V_0}{4\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}$ for example, if $\begin{matrix}b=a\\c=2a\end{matrix}$ then $q_2=2q_0$

  in the case, the nonzero E-field between b and c, and builds up the potential at a

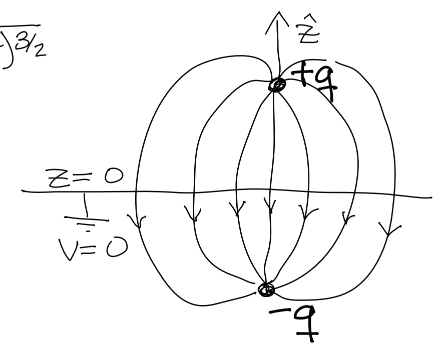
* Example 2: dipole: point charge $+q$ at $z=d$ and $-q$ at $z=-d$

$$V(z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}+\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

  ~ note that $V(z=0)=0$ so we can form a boundary value problem for $z>0$, $V(z=0)=0$ with the same solution!

  ~ induced surface charge: $\sigma=\varepsilon_0 E_n=-\varepsilon_0\frac{\partial V}{\partial n}=\frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$

  total induced charge:

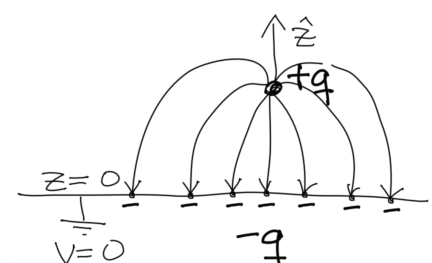


$$Q=\int\sigma da=\int_{\phi=0}^{2\pi}\int_{s=0}^{\infty}\sigma\, s\, ds\, d\phi=\frac{-2\pi qd}{2\pi}\int_{s=0}^{\infty}(s^2+d^2)^{-3/2}s\,ds$$

$$=-qd\int_{s=0}^{\infty}u^{-3/2}\tfrac{1}{2}du=qd\cdot -u^{-1/2}\Big|_{s=0}^{\infty}=-q \qquad \text{let } \begin{matrix}u=s^2+d^2\\du=2s\,ds\end{matrix}$$

  ~ force on the charge:

$$\vec{F}=q\vec{E}=-q\nabla V=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$



  ~ energy in the system: $W=\frac{1}{2}(W_0)=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$

  this is only half the value of dipole problem, because the induced charge is brought into zero potential (no work)