Section 3.3.2 - Separation of Variables (Spherical)

* same technique as in rectangular coordinates
  - ~ the differential equations are more complex, but we only solve them once
  - ~ boudnary conditions are of two types
    - a) radial - external boundary condition - treated in the same way as cartesian
    - b) angular - internal to the problem - almost always have the same solution
* key principles:
  - ~ separation of variables $\quad V(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$
  - ~ orthogonality of $\quad \Theta(\theta) = P_\ell(\cos\theta)$
  - ~ boundary conditions $\quad r\to 0,\ r=a,\ r\to\infty$

* separation of variables - slight twist: solve one eigenvalue at a time

$$\nabla^2 V(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}V + \frac{1}{r^2\sin^2\theta}\overbrace{\frac{\partial^2}{\partial\phi^2}V}^{-m^2V}}_{-\ell(\ell+1)V} = -\rho/\varepsilon_0 = 0$$

RADIAL EQUATION

$$\frac{d}{dr}r^2\frac{d}{dr}R(r) = \ell(\ell+1)R(r)$$

let $R(r)=r^\alpha$ $\quad \alpha(\alpha+1)=\ell(\ell+1)$

$\alpha = \ell\ ,\ -(\ell+1)$

$$R(r) = Ar^\ell + Br^{-\ell-1}$$

POLAR EQUATION ($m=0$)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\Theta(\theta) = -\ell(\ell+1)\Theta(\theta)$$

let $x=\cos(\theta)$ $\quad dx=-\sin\theta\,d\theta$

$\Theta(\theta)=P_\ell(x)$ $\quad \sin\theta\,d\theta\,d\phi \to -dx\,d\phi$

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_\ell(x) + \ell(\ell+1)P_\ell(x) = 0$$

$\Theta(\theta) = P_\ell(x) = P_\ell(\cos\theta)$; $Q_\ell(\cos\theta)$ diverges

AZIMUTHAL EQ.

$$\frac{d^2}{d\phi^2}\Phi = -m^2\Phi$$

$\Phi(\phi) = e^{im\phi}$

if $m=0$ then

$\Phi(\phi) = const$

* general solution

$$\nabla^2 V = 0 \qquad V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

* boundary conditions
  - i) at $r=0$, $\frac{1}{r^{\ell+1}}\to\infty$ so $B_\ell=0$
  - ii) at $r=\infty$, $r^\ell\to\infty$ so $A_\ell=0$
  - iii) at $r=a$, (1) $V_0(\theta) = V(a,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right)P_\ell(\cos\theta)$ $\qquad E_{ext} = E_0\hat{x} = -\nabla(-r^1\cos\theta)$

    (2) $\frac{\partial V_0}{\partial r}(\theta) = \frac{\partial V}{\partial r}(a,\theta) = \sum_{\ell=0}^{\infty}\left(\ell A_\ell a^{\ell-1} - (\ell+1)\frac{B_\ell}{a^{\ell+2}}\right)P_\ell(\cos\theta)$

  surface boundary at the interface between two regions with surface charge σ

$$\begin{matrix}\nabla\cdot\varepsilon_0\vec{E}=\rho \\ \nabla\times\vec{E}=0\end{matrix} \Rightarrow \begin{matrix}\hat{n}\cdot(\vec{E}_2-\vec{E}_1)=\sigma/\varepsilon_0 \\ \hat{n}\times(\vec{E}_2-\vec{E}_1)=0\end{matrix} \qquad \begin{matrix}E_{2n}-E_{1n}=\sigma/\varepsilon_0 \\ E_{2t}-E_{1t}=0\end{matrix} \Rightarrow \begin{matrix}V_1'(a)-V_2'(a)=\sigma/\varepsilon_0 \\ V_1(a)=V_2(a)\end{matrix}$$

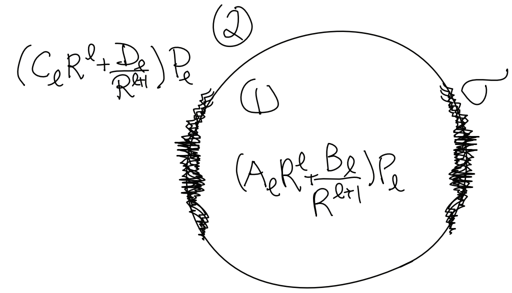
* properties of the Legendre polynomials
  - ~ Rodrigues formula $\quad P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell \qquad \ell=0,1,2,\ldots$
  - ~ orthogonality

$$\langle P_\ell|P_{\ell'}\rangle \equiv \int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx = \int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \begin{cases}0 & \text{if } \ell'\neq\ell \\ \frac{2}{2\ell+1} & \text{if } \ell'=\ell\end{cases}$$

  - ~ this is only one independent solution
    the other solutions $Q(x)$ blows up at the N&S poles $(\theta=0,2\pi)$
    and doesn't satisfy continuity boundary conditions

Problem 3.9

* spherical shell of charge $\sigma = \sigma_0 \sin^2\theta$



inside region: $V_1(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

outside region: $V_2(r,\theta) = \sum_{\ell=0}^{\infty}\left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

$4\times\infty$ unknowns
4 B.C.'s.

boundary conditions:

i) $V_1(0,\theta)$ finite $\Longrightarrow B_\ell = 0$

ii) $V_2(\infty,\theta)$ finite $\Longrightarrow C_\ell = 0$ (let $C_0 = 0$ also)

iii) $V_1(R,\theta) = V_2(R,\theta)$ $\quad \sum_{\ell=0}^{\infty}(A_\ell R^\ell + 0)P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty}\left(0 + \frac{D_\ell}{R^{\ell+1}}\right)P_\ell$

$$\sum_{\ell=0}^{\infty}\left(A_\ell R^\ell - \frac{D_\ell}{R^{\ell+1}}\right)P_\ell(\cos\theta) = 0 \Longrightarrow D_\ell = A_\ell R^{2\ell+1}$$

iv) $E_{2n} - E_{1n} = \sigma/\varepsilon_0$

$$-\frac{\partial V_2}{\partial r}\Big|_R + \frac{\partial V_1}{\partial r}\Big|_R = \frac{\sigma}{\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$$

$$\sum_{\ell=0}^{\infty}\left(D_\ell\frac{(\ell+1)}{R^{\ell+2}} + A_\ell \cdot \ell R^{\ell-1}\right)P_\ell(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$$

$\sin^2\theta = 1 - \cos^2\theta$
$= -\cos^2\theta + \frac{1}{3} + \frac{2}{3}$
$= -\frac{2}{3}P_2(\cos\theta) + \frac{2}{3}P_0(\cos\theta)$

$$* \quad \sum_{\ell=0}^{\infty} A_\ell(2\ell+1)R^{\ell-1}\cdot P_\ell(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\sin^2\theta$$

$$(A_0 R^{-1})P_0 + (A_1 \cdot 3R^0)P_1 + (A_2 \cdot 5R)P_2 + \ldots = \left(\frac{\sigma_0}{\varepsilon_0}\frac{2}{3}\right)P_0 + 0 + \left(\frac{\sigma_0}{\varepsilon_0}\cdot\frac{-2}{3}\right)P_2 + \ldots$$

$$A_0 = \frac{2\sigma_0 R}{3\varepsilon_0} \quad A_1 = 0 \quad A_2 = \frac{-2\sigma_0}{15\varepsilon_0 R}$$

solutions: inside $\quad V_1 = \frac{+2\sigma_0}{3\varepsilon_0}\left(R - \frac{r^2}{5R}\frac{1}{2}(3\cos^2\theta - 1)\right)$ $\qquad V_1 = V_2 \;\; @\; r = R$

outside $\quad V_2 = \frac{+2}{3}\frac{\sigma_0}{\varepsilon_0}\left(\frac{R^2}{r} - \frac{R^4}{5r^3}\frac{1}{2}(3\cos^2\theta - 1)\right)$ $\qquad -V_2' + V_1' = \sigma/\varepsilon_0 \;\; @\; r = R$

alternate solution of B.C. iv (use integrals to extract components like in Section 3.2.1)

$$* \quad \int_{\theta=0}^{\pi} P_0(\cos\theta)\cdot\sin^2\theta\,\sin\theta\,d\theta = \int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}$$

$$\int_{\theta=0}^{\pi} P_0(\cos\theta)\cdot P_0(\cos\theta)\,\sin\theta\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = \frac{2}{1}$$

$$\int_0^{\pi} P_1(\cos\theta)\cdot\sin^2\theta\,\sin\theta\,d\theta = \int_0^{\pi}\cos\theta\cdot\sin^3\theta\,d\theta = 0$$

$$\int_0^{\pi} P_1(\cos\theta)\cdot P_1(\cos\theta)\,\sin\theta\,d\theta = \int_0^{\pi}\cos^2\theta\cdot\sin\theta\,d\theta = \frac{2}{3}$$

$$\int_0^{\pi} P_2(\cos\theta)\cdot\sin^2\theta\,\sin\theta\,d\theta = \int_0^{\pi}\frac{1}{2}(3\cos^2\theta - 1)\cdot\sin^3\theta\,d\theta = \frac{-4}{15}$$

$$\int_0^{\pi} P_2(\cos\theta)\cdot P_2(\cos\theta)\,\sin\theta\,d\theta = \int_0^{\pi}\frac{1}{4}(3\cos^2\theta - 1)^2\cdot\sin\theta\,d\theta = \frac{2}{5}$$