

Section 3.4 - Multipoles

* binomial expansion

$$(a+b)^0 = 1$$

$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

~ general form

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}$$

~ if $n \rightarrow \alpha$ (any real number), then the series does not terminate

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

~ example: $\frac{1}{1-x} = 1 - (-x) + \frac{-1 \cdot -2}{1 \cdot 2} (-x)^2 + \frac{-1 \cdot -2 \cdot -3}{1 \cdot 2 \cdot 3} (-x)^3 + \dots$

$$= 1 + x + x^2 + x^3 + \dots \quad \text{for radius of convergence } |x| < 1$$

* Pascal's triangle

n=0					
n=1		1			
n=2		1	2	1	
n=3		1	3	3	1
n=4	1	4	6	4	1

$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

* 2-pole expansion

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_+} - \frac{q}{r_-} \right)$$

$$r_{\pm} = r \mp \frac{1}{2}d$$

$$r_{\pm}^2 = r^2 \mp r d \cos \theta + \frac{1}{4}d^2$$

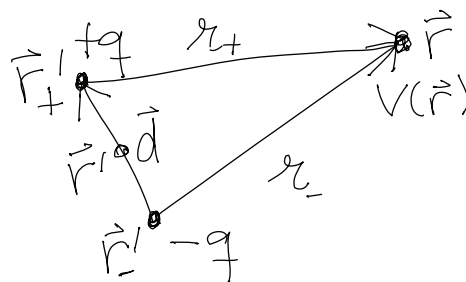
$$\frac{1}{r_{\pm}} \approx \frac{1}{r} \left(1 \mp \frac{d}{2r} \cos \theta \right)^{\pm 1}$$

$$\approx \frac{1}{r} \left(1 \pm \frac{d}{2r} \cos \theta + \dots \right)$$

$$V(\vec{r}) = \frac{q d \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

$$\boxed{\vec{p} = q\vec{d}}$$

electric dipole moment



* general axial-symmetric multipole expansion

$$r^2 = (\vec{r} - \vec{r}')^2 = r^2 (1 - 2 \frac{r'}{r} \cos \tau + (\frac{r'}{r})^2) \equiv r^2 (1 + \epsilon)$$

$$\frac{1}{r} = \frac{1}{r} (1 + \epsilon)^{-1/2} = \frac{1}{r} \left(1 - \frac{1}{2} \epsilon + \frac{3}{8} \epsilon^2 - \frac{5}{16} \epsilon^3 + \dots \right)$$

$$= \frac{1}{r} \left(1 + \frac{r'}{r} \cos \tau + \frac{r'^2}{r^2} \frac{1}{2} (3 \cos^2 \tau - 1) + \frac{r'^3}{r^3} \frac{1}{2} (5 \cos^3 \tau - 3 \cos \tau) + \dots \right)$$

$$= \frac{1}{r} \left(P_0(\cos \tau) + \frac{r'}{r} P_1(\cos \tau) + \frac{r'^2}{r^2} P_2(\cos \tau) + \dots \right) = \boxed{\sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) P_{\ell}(\cos \theta)}$$

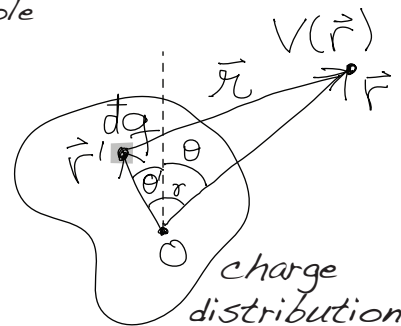
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} P_{\ell}(\cos \theta) \underbrace{\int d\tau r'^{\ell} P_{\ell}(\cos \theta)}_{Q_{int}^{(\ell)}}$$

multipole potential

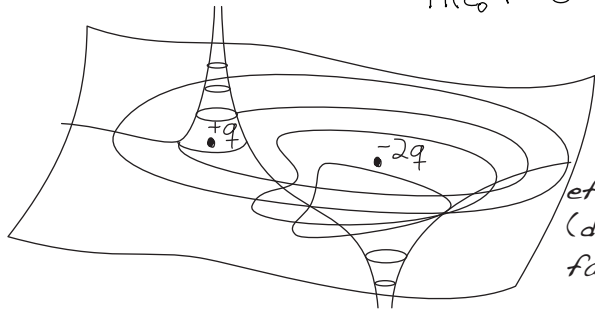
$Q_{int}^{(\ell)}$

electric multipole (monopole, dipole, quadrupole)

~ $Q_{int}^{(\ell)}$ are coefficients of the general solution of Laplace equation in spherical coords

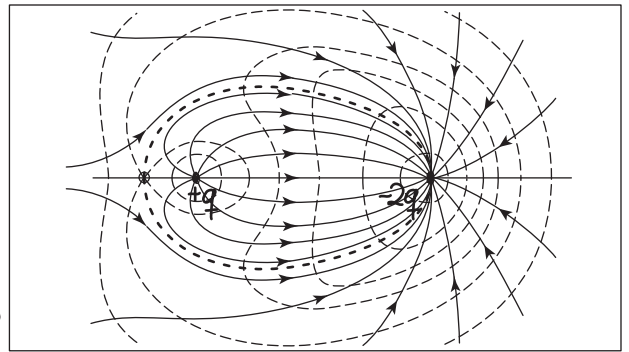


* monopole $V(\vec{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int dq' = \frac{q}{4\pi\epsilon_0 r}$



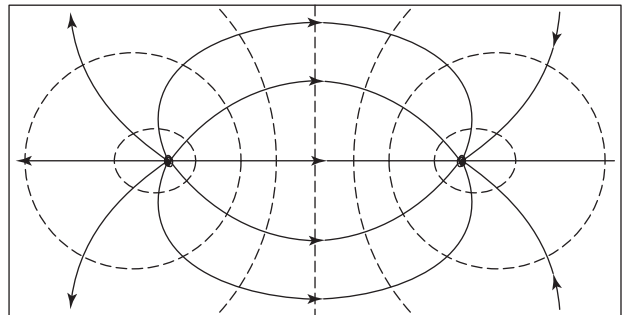
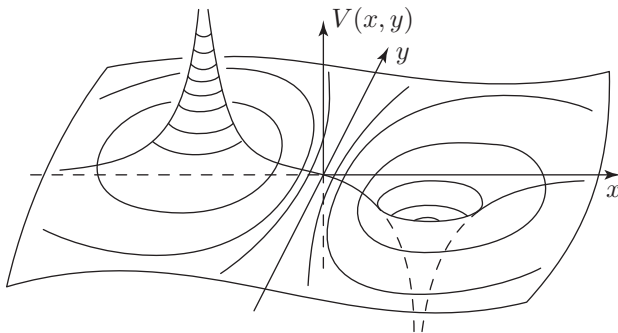
$$q \equiv \int dq'$$

effective monopole
(dominated by $-2q$
far from the origin)



* dipole $V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int dq' r' \cos\theta = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$ $\vec{p} = \int dq' \vec{r}'$

if $q = \int dq' = 0$ then $T_{\vec{a}}[\vec{p}] = \int dq' (\vec{r}' - \vec{a}) = \int dq' \vec{r}' - \vec{a} \int dq' = \vec{p}$



* quadrupole

$$V_2(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int dq' r'^2 \frac{1}{2} (3\cos^2\theta - 1) = \frac{1}{4\pi\epsilon_0 r^5} \int dq' \frac{1}{2} (3(\vec{r}' \cdot \vec{r})^2 - r'^2 r^2)$$

$$Q_{xx} + Q_{yy} + Q_{zz} = 0$$

$$= \frac{1}{2} \frac{\vec{r} \cdot \vec{Q} \cdot \vec{r}}{4\pi\epsilon_0 r^5}$$

$$\vec{Q} = \int dq' (3\vec{r}'\vec{r}' - r'^2 \mathbf{I}) = \int dq' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3yx' & 3y^2 - r'^2 & 3y'z' \\ 3zx' & 3zy' & 3z^2 - r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq' (3r'_i r'_j - \delta_{ij} r'^2)$$

$$T_{\vec{a}}[\vec{Q}] = \int dq' 3(\vec{r}' - \vec{a})(\vec{r}' - \vec{a}) - (r' - a)^2 \mathbf{I}$$

$$= \int dq' (3\vec{r}'\vec{r}' - r'^2 \mathbf{I}) - 3(\vec{r}'\vec{a} + \vec{a}\vec{r}' - \vec{a}\vec{a}) + (2\vec{r}' \cdot \vec{a} + a^2) \mathbf{I}$$

$$= \vec{Q} - [3(\vec{p}\vec{a} + \vec{a}\vec{p}) - 2\vec{p}\vec{a}\mathbf{I}] + [3\vec{a}\vec{a} - a^2 \mathbf{I}] q$$

$$\vec{Q} = \sum_i 3(\vec{p}_i \cdot \vec{a}_i + \vec{a}_i \cdot \vec{p}_i) - 2\vec{p}_i \cdot \vec{a}_i \mathbf{I} \quad \text{dipoles } \vec{p}_i \text{ at positions } \vec{a}_i$$

