Section 4.4.1 - Linear Dielectrics

* going from polarizability (α) to susceptibility (χ_e)

~ atoms: $\vec{p} = \alpha \vec{E}$

~ material: $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\Delta \vec{p}}{\Delta \tau} = \frac{\Delta n}{\Delta \tau} \vec{p} = N \alpha \vec{E}$ $\quad \boxed{\varepsilon_0 \chi_e \cong N\alpha}$ (See HW9 for refinements)

* material properties:

~ linear: χ_e is independent of field magnitude $|\vec{E}|$

~ isotropic: χ_e is a scalar (independent of direction $\hat{E}$)

~ homogeneous: χ_e is independent of position

~ nonisotropic material:

$\vec{P} = \varepsilon_0 \overleftrightarrow{\chi}_e \vec{E}$ (like $\overleftrightarrow{\alpha}$)

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

* permittivity: absolute $\varepsilon = \varepsilon_0 \varepsilon_r$, relative $\varepsilon_r = K$ (dielectric const.)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \underbrace{(1+\chi_e)}_{\varepsilon_r} \vec{E} = \underbrace{\varepsilon_0 \varepsilon_r}_{\varepsilon} \vec{E} = \varepsilon \vec{E}$$

~ constitutive eq: $\boxed{\vec{D} = \varepsilon \vec{E}}$

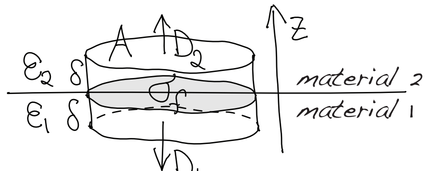
others: $\vec{B} = \mu \vec{H} \quad \vec{J} = \sigma \vec{E}$

~ property of the material: $\varepsilon_r = 1 + \chi_e = \varepsilon / \varepsilon_0$

* continuity boundary conditions

a) FLUX

$\boxed{\nabla \cdot \vec{D} = \rho_f}$



~ Gaussian pillbox

$$\Phi_D = \hat{z} \cdot D_2 A - \hat{z} \cdot D_1 A = \sigma_f A = Q_f$$

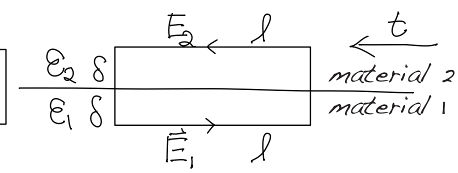
~ Integration of $\nabla \cdot \vec{D} = \rho_f$ across boundary

$$\lim_{\delta \to 0} \int_{-\delta}^{\delta} dz \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \int_{-\delta}^{\delta} \sigma_f \, \delta(z-z') \, dz$$

$$\int_{-\delta}^{\delta} dD_z = \boxed{\hat{n} \cdot \Delta \vec{D} = \sigma_f} \quad \boxed{-\varepsilon_2 \frac{\partial V_2}{\partial n} + \varepsilon_1 \frac{\partial V_2}{\partial n} = \sigma_f}$$

b) FLOW

$\boxed{\nabla \times \vec{E} = 0}$



~ Amperian loop

$$\mathcal{E}_E = \hat{t} \cdot \vec{E}_2 l - \hat{t} \cdot \vec{E}_1 l = 0$$

~ Integration of $\nabla \times \vec{E} = 0$ across boundary

$$\lim_{\delta \to 0} \int_{-\delta}^{\delta} dz \, \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{y} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0$$

$$= \int_{-\delta}^{\delta} \hat{x} \, dE_y - \hat{y} \, dE_x = \boxed{\hat{n} \times \Delta \vec{E} = 0} \quad \boxed{V_2 = V_1}$$

~ the only difference in dielectric boundary value problems is $\varepsilon_1, \varepsilon_2$ in boundary cond.

* example 4.7: dielectric ball in electric field

$$V_1 = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos\theta)$$

$$V_2 = \sum_{l=0}^{\infty} \left(C_l r^l + D_l r^{-l-1} \right) P_l(\cos\theta)$$

$\lim r \to 0 \; V_1(r) \neq \infty \quad B_l = 0$

$\lim r \to \infty \; V_2(r) = -E_0 r \cos\theta \quad C_l = -E_0 \delta_{l1}$

$V_1(R) = V_2(R) \quad A_l R^l = C_l R^l + D_l R^{-l-1}$

$-\varepsilon_2 V_2'(R) + \varepsilon_1 V_1'(R) = \sigma_f = 0$

$-\varepsilon_2 \left(C_l \cdot l R^{l-1} + D_l (-l-1) R^{-l-2} \right) + \varepsilon_1 \left(A_l l R^{l-1} \right) = 0$

if $l \neq 1 \quad D_l = A_l R^{2l+1} \quad D_l = A_l = 0$

if $l = 1 \quad A_1 = -E_0 + D_1 R^{-3}$

$- \varepsilon_2 \left(-E_0 - 2 D_1 R^{-3} \right) + \varepsilon_1 A_1 = 0$

$- \varepsilon_2 \left(-E_0 - 2 (A_1 + E_0) \right) + \varepsilon_1 A_1 = 0$

$3 \varepsilon_2 E_0 + (\varepsilon_1 + 2\varepsilon_2) A_1 = 0$

$$A_1 = \frac{-3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} E_0$$

if $\varepsilon_1 = \varepsilon_r \varepsilon_2$, $A_1 = \frac{-3}{\varepsilon + 2} E_0$