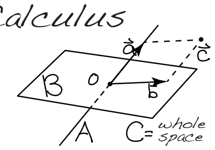
Review of Electrostatics (Chapters 1-4)

* Chapter 1: Mathematics - Vector Calculus

- ~ Vectors (it's all about being Linear!)
  linear combinations; projections
  basis (independence, closure)
- ~ Metric & Cross Product (Bilinear)
  orthonormal basis $\hat{n}\cdot\hat{n} = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times$
  longitudinal / transverse projections
- ~ Linear Operators
  eigenstuff: rotations / stretches
- ~ Function Spaces - continuous vs discrete
  Sturm-Liouville (orthogonal eigenfunctions)
- ~ Vector Derivatives and Integrals (linearization)
  Differentials ordered naturally by dimension



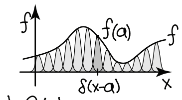
$$\vec{x} = \vec{b}_i x^i \equiv \sum_{i=1}^{3} \vec{b}_i x^i$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$\sum_{i=1}^{3} \hat{e}_i \hat{e}_i \cdot = \mathbb{I}$$

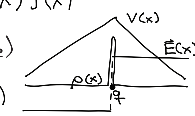
STOKES
$$\oint_{\partial R} \omega = \int_R d\omega$$

- ~ Delta function (pole)



$$a_i = \sum_{j=1}^{n} \delta_{ij} a_j \qquad f(x) = \int_{-\infty}^{\infty} dx' \delta(x-x') f(x')$$

Greens function (tent/pole)



$$\nabla^2 G(\vec{r}) = \nabla^2 \frac{-1}{4\pi r} = \nabla\cdot\frac{\hat{r}}{4\pi r^2} = \delta^3(\vec{r})$$

- ~ Poincare: potentials & derivative chain
  $d\omega = 0 \Leftrightarrow \omega = d\alpha \qquad \nabla\times\vec{E} = \vec{0} \leftrightarrow \vec{E} = -\nabla V \qquad \nabla\cdot\vec{B} = 0 \leftrightarrow \vec{B} = \nabla\times\vec{A}$
- ~ Helmholtz theorem: source and potential

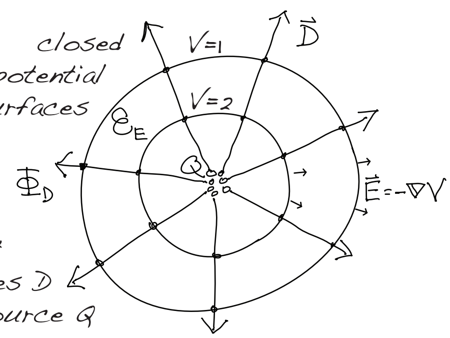
$$\vec{F} = -\nabla\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho}\right)}_{V} + \nabla\times\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}\right)}_{\vec{A}}$$

$$= -\nabla V + \nabla\times\vec{A} \qquad \nabla^2(V, \vec{A}) = -(\rho, \vec{J}) \qquad \begin{matrix}\nabla\cdot\vec{F} = \rho \\ \nabla\times\vec{F} = \vec{J}\end{matrix}$$

RANK	REGION	INTEGRAL		THEOREM	DERIVATIVE	GEOMETRY
scalar	∂ Q point	$\Delta f = f\vert_a^b$	change	$f\vert_a^b = \int_a^b \nabla f\cdot d\vec{l}$	$df = \nabla f \cdot d\vec{l}$	level surface d
vector	∂ P path	$\mathcal{E}_A = \oint_{\partial S}\vec{A}\cdot d\vec{l}$	flow	$\oint_{\partial S}\vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{a}$	$d\vec{A}\cdot d\vec{l} = \nabla\times\vec{A}\cdot d\vec{a}$	flow sheets d
p-vector	∂ S surface	$\Phi_B = \oint_{\partial V}\vec{B}\cdot d\vec{a}$	flux	$\oint_{\partial V}\vec{B}\cdot d\vec{a} = \int_V \nabla\cdot\vec{B}\,d\tau$	$d\vec{B}\cdot d\vec{a} = \nabla\cdot\vec{B}\,d\tau$	flux tubes d
p-scalar	V volume	$Q_\rho = \int_V \rho\,d\tau$	subst.	×	×	subst boxes d

* Chapter 2: Formulations of Electrostatics

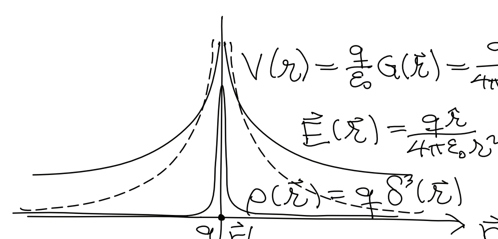
Integral	Differential	Boundary
$\vec{E} = \int \frac{dq'\,\hat{r}}{4\pi\varepsilon_0 r^2}$	$\nabla^2\vec{E} = \nabla\rho/\varepsilon_0$	$dq = \sigma\,da$
$\mathcal{E}_E = 0$	$\nabla\times\vec{E} = 0$	$\Delta\hat{n}\times\vec{E} = E_{2t} - E_{1t} = 0$
$\Phi_D = Q$	$\nabla\cdot\vec{D} = \rho$	$\Delta\hat{n}\cdot\vec{D} = D_{2n} - D_{1n} = \sigma_f$
$V = -\int \vec{E}\cdot d\vec{l}$	$\vec{E} = -\nabla V$	$V_2 - V_1 = 0$
$V = \int \frac{dq'}{4\pi\varepsilon_0 r}$	$\nabla^2 V = -\rho/\varepsilon_0$	$-\varepsilon_2\partial_n V_2 + \varepsilon_1\partial_n V_1 = \sigma_f$



Relation between potential, field, and source:

$$V \underset{-\int\vec{E}\cdot d\vec{l}}{\overset{-\nabla V}{\rightleftarrows}} \vec{E} \underset{\int\frac{dq'\,\hat{r}}{4\pi\varepsilon_0 r^2}}{\overset{\nabla\cdot\varepsilon_0\vec{E}}{\rightleftarrows}} \rho \quad dq' = \rho(\vec{r}')\,d\tau'$$

$$\int\frac{dq'\,\hat{r}}{4\pi\varepsilon_0 r^2} \qquad \underset{\int}{\overset{d}{\rightleftarrows}} \qquad -\varepsilon_0\nabla^2 V$$



$$V(\vec{r}) = \frac{q}{\varepsilon_0} G(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}$$

$$\vec{E}(\vec{r}) = \frac{q\,\hat{r}}{4\pi\varepsilon_0 r^2}$$

$$\rho(\vec{r}) = q\,\delta^3(\vec{r})$$

- ~ Work and Electric field energy: flux × flow

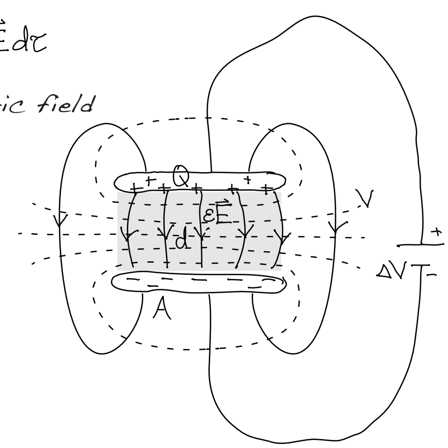
$\vec{F} = q\vec{E} \qquad \vec{F} = m\vec{g}$

$W = q\underbrace{Ed}_{\text{potential}=V} \qquad W = m\underbrace{gh}_{\text{potential danger}}$

$$\Delta W = \int \Delta\rho_f V\,d\tau = \oint d\vec{a}\cdot(\Delta\vec{D}\,V) - \int\Delta\vec{D}\cdot\vec{E}\,d\tau$$

$$W = \int \tfrac{1}{2}\vec{D}\cdot\vec{E}\,d\tau \quad \text{energy density of electric field}$$

- ~ Conductors and Capacitance: flux / flow
  $\rho = E = 0$, V=const inside; $D = \sigma$, V laminar outside conductor



$$C = \frac{Q}{\Delta V} = \frac{\Phi_D}{\mathcal{E}_E} \qquad Q = \oint d\vec{a}\cdot\vec{D} = \Phi_D \text{ (closed surface)}$$

$$= \frac{\varepsilon\Phi_E}{\mathcal{E}_E} \approx \frac{\varepsilon A}{d} \qquad \Delta V = \int d\vec{l}\cdot\vec{E} = \mathcal{E}_E \text{ (open path)}$$

* Chapter 3: Solutions of LaPlace Equation

~ Uniqueness Theorem for exterior boundary conditions

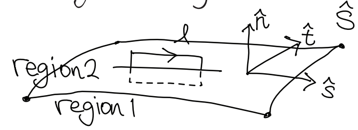
$$0 = \int_{\partial V} da\, \underbrace{U}_{(a)} \underbrace{\frac{\partial U}{\partial n}}_{(b)} = \oint_{\partial V} d\vec{a} \cdot (U \nabla U) = \int_V \nabla \cdot (U \nabla U)\, d\tau = \int_V U \nabla^2 U + (\nabla U)^2\, d\tau$$

a) Dirichlet B.C. specifies potential on boundary; b) Neuman B.C. specifies flux on boundary

~ continuity boundary conditions stitch potentials together in adjacent regions

Flux: $\vec{D} \equiv \varepsilon \vec{E}$

Flow:



$\hat{s} \times \hat{t} = \hat{n}$

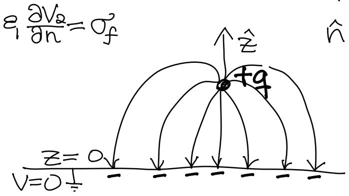
$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma \qquad -\varepsilon_2 \frac{\partial V_2}{\partial n} + \varepsilon_1 \frac{\partial V_2}{\partial n} = \sigma_f \qquad \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \qquad V_2 = V_1$

A) METHOD OF IMAGES

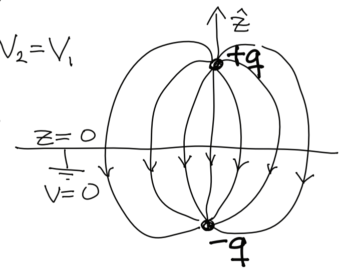
find a point charge distribution

with the same B.C.'s

same solution by uniques theorem



is equivalent to



B) METHOD OF SEPARATION OF VARIABLES

separate Laplacian (10 known coordinate systems)

solve Sturm-Louiville ODE in each dimension

match boundary conditions to find coefficients

Fourier trick : orthogonal basis functions

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{\vec{k}\cdot\vec{r}} = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin(k_n y)$$

$$\phi_n(x) = \sin(k_n x) \qquad V(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\langle \phi_n | \phi_m \rangle = \int_0^a \sin(k_n x) \cdot \sin(k_m x)\, dx = \frac{a}{2} \delta_{nm}$$

$$C_m = \langle \phi_m(x) | V(x) \rangle \big/ \frac{a}{2}$$

C) METHOD OF MULTIPOLE MOMENTS

series expansion of potential about origin or infinity

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \int dq'\, r' \cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \vec{p} = \int dq'\, \vec{r}'$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \int dq'\, r'^2 \tfrac{1}{2}(3\cos^2\theta - 1) = \frac{1}{4\pi\varepsilon_0 r^5} \int dq'\, \tfrac{1}{2}\left(3(\vec{r}\cdot\vec{r}')^2 - r^2 r'^2\right)$$

$$= \frac{\frac{1}{2}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}{4\pi\varepsilon_0 r^5} \qquad \overleftrightarrow{Q} = \int dq' (3\vec{r}'\vec{r}' - I r'^2) = \int dq' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2 - r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2 - r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq' (3 r_i' r_j' - \delta_{ij} r'^2)$$

$$Q_{ext}^{(\ell)} = \frac{A_\ell}{4\pi\varepsilon_0} = \int dq' \frac{1}{r'^{\ell+1}} P_\ell(\cos\theta')$$

$$Q_{int}^{(\ell)} = \frac{B_\ell}{4\pi\varepsilon_0} = \int dq'\, r'^\ell\, P_\ell(\cos\theta')$$

* Chapter 4: Dielectric Materials - Dipole

$\vec{N} = \vec{p} \times \vec{E} \qquad U = -\vec{p}\cdot\vec{E} \qquad \vec{F} = \nabla(\vec{p}\cdot\vec{E})$

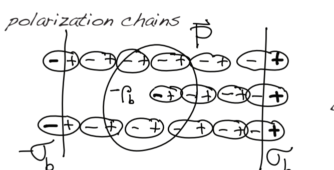
$$\vec{p} = \alpha \vec{E} \qquad \varepsilon_0 \chi_e \cong N\alpha$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\Delta\vec{p}}{\Delta\tau} = \frac{\Delta n}{\Delta\tau} \vec{p} = N\alpha\vec{E}$$

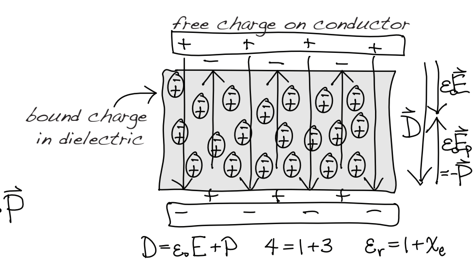
$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \underbrace{(1+\chi_e)}_{\varepsilon_r} \vec{E} = \underbrace{\varepsilon_0 \varepsilon_r}_{\varepsilon} \vec{E} = \varepsilon \vec{E}$$

$$\varepsilon_r = 1 + \chi_e = \varepsilon / \varepsilon_0$$

$$\nabla \cdot \vec{D} = \nabla \cdot \varepsilon_0 \vec{E} + \nabla \cdot \vec{P} = \rho - \rho_b = \rho_f$$



$\sigma_b = \vec{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\vec{P}$



$D = \varepsilon_0 E + P \qquad 4 = 1 + 3 \qquad \varepsilon_r = 1 + \chi_e$

* Outlook - road to electrodynamic equations

$$\begin{array}{ccccccccc} \lambda & \xrightarrow{d} & (V,\vec{A}) & \xrightarrow{d} & (\vec{E},\vec{B}) & \xrightarrow{d} & 0 & & \\ & & & & \varepsilon \Updownarrow \mu & \nwarrow \sigma, \vec{F} & & & \\ & & (\vec{C},U) & \xrightarrow{d} & (\vec{D},\vec{H}) & \xrightarrow{d} & (\rho, J) & \xrightarrow{d} & 0 \\ (0) & & (1) & & (2) & & (3) & & (4) \end{array}$$

$$\Phi_D = Q_{encl} \qquad \Phi_B = 0 \qquad -\Box^2 (V,\vec{A}) = (\rho/\varepsilon, \mu\vec{J})$$

$$\mathcal{E}_E = -\frac{\partial\Phi_B}{\partial t} \qquad \mathcal{E}_H = I_{encl} + \frac{\partial\Phi_D}{\partial t} \qquad \text{(wave equation)}$$

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = (\rho\vec{E} + \vec{J}\times\vec{B})\, d\tau$	Lorentz force
$\partial_t \rho + \nabla\cdot\vec{J} = 0$	Continuity
$\nabla\cdot\vec{D} = \rho \qquad \nabla\times\vec{E} + \partial_t\vec{B} = \vec{0}$	Maxwell electric,
$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{H} - \partial_t\vec{D} = \vec{J}$	magnetic fields
$\vec{D} = \varepsilon\vec{E} \qquad \vec{J} = \sigma\vec{E} \qquad \vec{B} = \mu\vec{H}$	Constitution
$\vec{E} = -\nabla V - \partial_t \vec{A} \qquad \vec{B} = \nabla\times\vec{A}$	Potentials
$V \to V - \partial_t \lambda \qquad \vec{A} \to \vec{A} + \nabla\lambda$	Gauge transform