University of Kentucky, Physics 416G
Problem Set #1, Rev. A, due Friday, 2014-09-05

1. a) Show graphically that the following equations define a set of points $\{\boldsymbol{x}\}$ on a **line** or **plane**,

	relational	parametric
line	$\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$	$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha$
plane	$\boldsymbol{A} \cdot \boldsymbol{x} = D$	$\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma$

where $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$ are constants, $\boldsymbol{x}_1$ is a fixed point on the line, $\boldsymbol{x}_2$ is a fixed point on the plane, $\boldsymbol{x}$ is an arbitrary point on the line or plane, and α, β, γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane).

Technically, the set of points $\{\boldsymbol{x} \mid \boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}\}$ or $\{\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha \mid \alpha \in \mathbb{R}\}$ each form a line, while the sets $\{\boldsymbol{x} \mid \boldsymbol{A} \cdot \boldsymbol{x} = D\}$ or $\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma \mid \beta, \gamma \in \mathbb{R}\}$ each form a plane.

b) What constraint between $\boldsymbol{a}$ and $\boldsymbol{d}$ is implicit in the formula $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$? Likewise, what relations between $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{A}$ must be satisfied if both equations describe the same plane?

c) For the line and the plane, substitute $\boldsymbol{x}$ from the parametric form into the relational form to show that they are consistent. What are $\boldsymbol{d}$ and D in terms of $\boldsymbol{a}, \boldsymbol{x}_1$ and $\boldsymbol{A}, \boldsymbol{x}_2$, respectively?

d) Let us define $\tilde{\boldsymbol{a}} \equiv \boldsymbol{A}/(\boldsymbol{a}\cdot\boldsymbol{A})$. It is parallel to $\boldsymbol{A}$ but *normalized* in the sense that $\boldsymbol{a} \cdot \tilde{\boldsymbol{a}} = 1$. Using the BAC-CAB rule, show that $\boldsymbol{x} = \boldsymbol{a}(\tilde{\boldsymbol{a}} \cdot \boldsymbol{x}) - \tilde{\boldsymbol{a}} \times (\boldsymbol{a} \times \boldsymbol{x})$ for any $\boldsymbol{x}$. This is a non-orthogonal projection of $\boldsymbol{x}$ into a vector parallel to the line and a vector parallel to the plane. Illustrate this projection and show which term corresponds to each.

e) Using d), calculate the point $\boldsymbol{x}_0$ at the intersection of the line and plane in terms of $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$.

f) Verify e) by showing $\boldsymbol{x}_0$ satisfies the relational equation for both the line and plane.

g) Let $\tilde{\boldsymbol{a}} = \frac{\boldsymbol{b}\times\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$, $\tilde{\boldsymbol{b}} = \frac{\boldsymbol{c}\times\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$, and $\tilde{\boldsymbol{c}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$. This definition of $\tilde{\boldsymbol{a}}$ is consistent with above if we require $\boldsymbol{A} = \boldsymbol{b}\times\boldsymbol{c}$. Calculate the nine combinations of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^T \cdot (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$. Note that $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is called the *dual* or *reciprocal* basis of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ because it is orthonormal to it in the above sense.

h) The *contravariant components* of $\boldsymbol{x}$ are defined as the components (α, β, γ) that satisfy the equation $\boldsymbol{x} = \boldsymbol{a}\alpha + \boldsymbol{b}\beta + \boldsymbol{c}\gamma$. In other words, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is the contravariant basis. Using $\boldsymbol{x} \cdot \tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\boldsymbol{x}$ in terms of dot products.

Bonus: How is this equivalent to the method of finding components by matrix inversion, as in the class notes, first paragraph?

i) Find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of $\boldsymbol{x}$. As above, they are defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy the equation $\boldsymbol{x} = \tilde{\boldsymbol{a}}\tilde{\alpha} + \tilde{\boldsymbol{b}}\tilde{\beta} + \tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the covariant basis.

Bonus: How does everything simplify if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ are an orthonormal basis?

2. Clifford algebra. The complete 3-vector algebra including dot and cross products can be implemented using the identity (I) as the unit scalar and Pauli matrices ($\boldsymbol{\sigma}$) as unit vectors:

$$1 = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\boldsymbol{x}} = \boldsymbol{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\boldsymbol{y}} = \boldsymbol{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\boldsymbol{z}} = \boldsymbol{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The dot and cross products can be represented by matrix multiplication, as in the formula

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = I(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) + i(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j) = I\delta_{ij} + i\epsilon_{ijk}\boldsymbol{\sigma}_k, \tag{2}$$

where the dot and cross products are interpreted in the usual sense of unit vectors. Note the difference between the imaginary i and the index $_i$. Often I is implicit, so that scalars are actually multiples of the identity matrix. This algebra generalizes to space-time, where the Dirac matrices γ_μ are used instead of the Pauli matrices.

a) Verify this formula for all nine products $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$. Which products are symmetric and which are antisymmetric? While in general, a product can have both symmetric and antisymmetric parts, this partition of symmetry into $i = j$ and $i \neq j$ is the defining feature of a Clifford algebra.

b) Show that any linear product $\boldsymbol{a} \circ \boldsymbol{b}$, can be decomposed into the sum $\boldsymbol{a} \circ \boldsymbol{b} = \{\boldsymbol{a} \circ \boldsymbol{b}\} + \langle \boldsymbol{a} \circ \boldsymbol{b} \rangle$ of symmetric $\{\boldsymbol{a} \circ \boldsymbol{b}\} \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} + \boldsymbol{b} \circ \boldsymbol{a})$ and antisymmetric $\langle \boldsymbol{a} \circ \boldsymbol{b} \rangle \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{b} \circ \boldsymbol{a})$ parts, with respect to exchange of $\boldsymbol{a}$ and $\boldsymbol{b}$. Show that $\langle \boldsymbol{a} \circ \boldsymbol{a} \rangle = 0$ always. Why are the diagonal elements of an antisymmetric matrix zero? Apply this decomposition to the product $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$.

c) The imaginary i in the above formula is not present in the ordinary cross product. It distinguishes [axial] pseudovectors from [polar] vectors. Using part a), calculate the value of the pseudoscalar $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k$ for all values of $i \neq j \neq k$. What is the dimension of this space? Use this result to distinguish between pseudoscalars and scalars. What is the analog of this triple product in terms of vector products?

Bonus: These matrices have the additional capability of adding scalars and vectors, which is not meaningful in basic vector spaces. Use this property to simultaneously solve the two equations $\boldsymbol{a} \cdot \boldsymbol{x} = D$ and $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$ from Problem #1 (with $\boldsymbol{A} = \boldsymbol{a}$) as a single matrix equation.

Also, Griffiths 3ed or 4ed chapter 1, problems #1, 4, 5, 8, 10, 12, 15, 16, 18.