University of Kentucky, Physics 416G
Problem Set #2, Rev. A, due Wednesday, 2014-09-17

1. Curvilinear coordinates: the *coordinate basis* $\boldsymbol{b}_i \equiv \partial\boldsymbol{r}/\partial q^i$ and *reciprocal basis* $\boldsymbol{b}^i \equiv \boldsymbol{\nabla} q^i = \partial q^i/\partial\boldsymbol{r}$ are the two natural dual bases to describe components of a vector field in curvilinear coordinates (q^1, q^2, q^3). For orthogonal systems, we use the single *orthonormal basis* $\hat{\boldsymbol{e}}_i = \boldsymbol{b}_i/h_i = \boldsymbol{b}^i h_i$, where $h_i = |\boldsymbol{b}_i| = 1/|\boldsymbol{b}^i|$ is a *scale factor*. Note that these basis vectors change from one point to the next. Calculate each of the following in both cylindrical and spherical coordinates.

a) Calculate $\boldsymbol{b}_i = (\boldsymbol{b}_s, \boldsymbol{b}_\phi, \boldsymbol{b}_z)$ as functions of $q^i = (s, \phi, z)$ and $(\boldsymbol{b}_r, \boldsymbol{b}_\theta, \boldsymbol{b}_\phi)$ as functions of (r, θ, ϕ).

b) Calculate the resulting scale factors h_i to get unit vectors in both coordinate systems. Write out the basis transformation matrix R for these vectors, i.e. $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})R$.

c) Construct the transformation matrices between unit bases, by considering rotations $R_z(\phi)$ (rotation by an angle ϕ about the z-axis) and $R_\phi(\theta)$ and compare with part b.

d) Calculate the reciprocal vectors $(\boldsymbol{b}^s, \boldsymbol{b}^\phi, \boldsymbol{b}^z)$ and $(\boldsymbol{b}^r, \boldsymbol{b}^\theta, \boldsymbol{b}^\phi)$. Show that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i{}^j$ in general.

e) Calculate the metric $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$ and reciprocal $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j$ in terms of h_i.

f) Calculate the general line element $\boldsymbol{dl} = \boldsymbol{b}_i dq^i$, area element $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl} \times \boldsymbol{dl}$, and volume element $d\tau = \frac{1}{6}\boldsymbol{dl} \cdot \boldsymbol{dl} \times \boldsymbol{dl}$ in cylindrical and spherical coorindates. These products are not zero but have 2 [or 6] identical terms, because differentials anticommute.

g) Apply the formulas $\nabla f = \frac{\hat{\boldsymbol{e}}_i}{h_i}\frac{\partial}{\partial q^i} f$, $\nabla \times \boldsymbol{A} = \frac{\hat{\boldsymbol{e}}_i}{h_j h_k}\frac{\partial}{\partial q^j} h_k A_k$, and $\nabla \cdot \boldsymbol{B} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial q^k} h_i h_j B_k$, for i, j, k cyclic, to cylindrical and spherical coordinates. Compare with Griffiths, inside front cover.

Bonus: Calculate the derivatives of the basis vectors along coordinate lines, $\boldsymbol{\Gamma}_{ij} = \Gamma_{ij}{}^k \boldsymbol{b}_k = \partial\boldsymbol{b}_i/\partial q^j = \partial^2\boldsymbol{r}/\partial q^i\, \partial q^j$. These *Christoffel symbols* and are needed to calculate derivative of vector fields in curvilinear coordinates.

2. The **Hodge dual operator** $*$ converts between scalar/vector and pseudo-scalar/vector differentials according to the definitions $*1 = dx\, dy\, dz = d\tau$, $\quad *d\tau = *dx\, dy\, dz = 1$, $\quad *dx^i = \frac{1}{2}\epsilon_{ijk} dx^j dx^k$, and $*dx^i dx^j = \epsilon_{ijk} dx^k$. The factor $\frac{1}{2}$ removes identical terms.

a) Calculate $(*dx, *dy, *dz)$ and $(*dy\, dz, *dz\, dx, *dx\, dy)$ to show that $*\boldsymbol{dl} = \boldsymbol{da}$ and $*\boldsymbol{da} = \boldsymbol{dl}$. What is the equivalent operation in HW1 #2?

b) The *codifferential operator* δ is defined by $\delta \equiv (-1)^p * d\, *$, where p is the dimension of the differential acted upon ($p = 0$ for scalars, $p = 1$ for $\boldsymbol{dl}$, $p = 2$ for $\boldsymbol{da}$, and $p = 3$ for $d\tau$). Show

i) $\delta f = 0$ for all scalar functions f,
ii) $\delta(\boldsymbol{A} \cdot \boldsymbol{dl}) = -\nabla \cdot \boldsymbol{A}$ for a vector field $\boldsymbol{A}$,
iii) $\delta(\boldsymbol{A} \cdot \boldsymbol{da}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{dl}$, and
iv) $\delta(f\, d\tau) = -(\nabla f) \cdot \boldsymbol{da}$.

Note that δ is the *adjoint* of d in that it switches between divergence and curl, etc.

c) Show that $-\nabla^2 f = -\delta d f = -(d\delta + \delta d) f$ for the Laplacian of a scalar function f, and likewise, $(\nabla^2 \boldsymbol{A}) \cdot \boldsymbol{dl} = -(d\delta + \delta d)(\boldsymbol{A} \cdot \boldsymbol{dl})$ for the Laplacian of a vector field. Note that by mixing differentials and codifferentials, it possible to generate nontrivial second derivatives. Interpret each [co]differential (d, δ) in terms of its corresponding vector derivative (∇).

Also, Griffiths 3ed[4ed] chapter 1, problems
#13[13], 19[20], 21[22], 25[26], 38[39], 39[40], 40[41], 42[43].