University of Kentucky, Physics 416G
Problem Set #3, Rev. B, due Friday, 2014-09-26

1. a) Plot a 2-D graph of the function $h(x, y) = 2xy$ by drawing level curves in the xy-plane.

b) Plot a 3-D graph of the 2-D function $h(x, y)$, ie. the surface $\{(x, y, z) \mid z = h(x, y)\}$.

c) Calculate the gradient of h at all points, and also the divergence and curl of the gradient. Plot the field lines and equipotentials of the gradient. Note that the equipotentials are curves, not surfaces, since $h(x, y)$ is 2-dimensional. Bonus: Calculate an equation for field line curves.

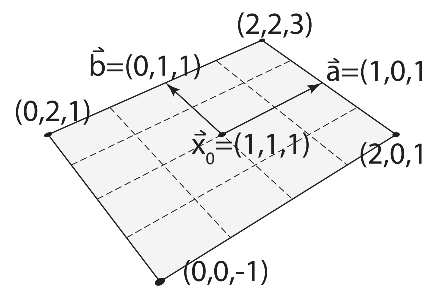
d) Construct a 3-D function $g(x, y, z)$ which has $z = h(x, y)$ as one level surface. Calculate the gradient of this function to obtain a normal vector to the surface $z = h(x, y)$ at all points on the surface. Compare your answer to the normal of the surface calculated using the cross product (hint: calculate $\boldsymbol{da}$ on the surface).

e) Show that there is a linear coordinate transformation $(x, y) \to (x', y')$ that transforms the function $h'(x', y') = y'^2 - x'^2$ into $h(x, y)$. We will find that both of these functions are $m = 2$ cylindrical harmonics.

f) Show that $h_1(x, y) = 2xy$ and $h_2(x, y) = y^2 - x^2$ form a basis for all functions which can be generated from h_1 and h_2 by rotating the coordinates (x, y) by an angle θ via the transformation $\boldsymbol{r}' = R_\theta\, \boldsymbol{r}$, i.e. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, where $\begin{matrix} c_\phi = \cos\phi \\ s_\phi = \sin\phi \end{matrix}$.

g) Verify Stokes' theorem $\oint_{\partial R} \boldsymbol{v} \cdot \boldsymbol{dl} = \int_R \boldsymbol{\nabla} \times \boldsymbol{v} \cdot \boldsymbol{da}$ using the vector field $\boldsymbol{v}(x, y, z) = \hat{\boldsymbol{x}} yz$ over the surface $z = h_1(x, y)$, where $x^2 + y^2 < 1$.

2. a) Integrate $\int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{da}$ where $\boldsymbol{v} = \hat{\boldsymbol{x}}\, x^2 + \hat{\boldsymbol{y}}\, 2yz + \hat{\boldsymbol{z}}\, xy$, and $\mathcal{S}$ is the parallelogram in the figure to the right. Hint: remember the parameterization of HW1 #1.



b) Integrate $\oint_{\partial\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{dl}$ along the boundary of $\mathcal{S}$.

c) Verify Stokes' theorem for the integral in part b.

2′ [alternate or bonus] Fundamental Theorem of Differentials: In class we discussed an extension of the Fundamental Theorem of Calculus in higher dimensions: $d \int_n \omega + \int_n d\omega = \omega$, where n is a 'radial' coordinate, and $\int_n$ is an indefinite integral starting at $n = 0$. This theorem encompases both the Poincaré lemmas (indefinite integral: potential theory), and the boundary theorems (definite integral: gradient, curl, divergence theorems).

a) Show that this theorem reduces to the FTC for both functions $f(x)$ and distributions $\rho(x)dx$ in one dimension. What happens to the constant of integration for indefinite integrals?

b) Apply this theorem to calculate the potential V of an conservative field $\boldsymbol{E}$, ie. $\boldsymbol{E} = -\nabla V$ when $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$. How does this relate to Stokes' theorem?

c) Apply this theorem to calculate the potential $\boldsymbol{A}$ of a solenoidal field $\boldsymbol{B}$, ie. $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ when $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Compare your answer with Griffiths 3ed[4ed] problem 5.51[53]. How does this relate [if at all] to Gauss' theorem?

Also, Griffiths 3ed[4ed] Ch. 1, #35[36], 44[45], 45[46], 46[47], 47[48], 48[49], 60[61], 61[62].