University of Kentucky, Physics 416G
Problem Set #4, Rev. A, due Friday, 2014-10-10

1. There are five equivalent **formulations of electrostatics**: (I) $\boldsymbol{E} = \int đq' \hat{\boldsymbol{\imath}} / 4\pi\epsilon_0 \imath^2$; (II) $\Phi_E = Q/\epsilon_0$, $\mathcal{E}_E = 0$; (III) $\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$, $\nabla\times\boldsymbol{E} = \boldsymbol{0}$; (IV) $V = \int đq'/4\pi\epsilon_0 \imath$; (V) $\nabla^2 V = -\rho/\epsilon_0$. In this exercise we explore connections between (I), (II), (III), and develop a new formulation (VI).

a) Show that (I) $\Rightarrow$ (II) by integrating $\boldsymbol{E}$ over an arbitrary closed surface and path.

b) Show that (I) $\Leftarrow$ (II) by the superposing the fields obtained by applying Gauss' law on point charge elements $đq'$ at each point $\boldsymbol{r}'$ in space.

c) Show that (I) $\Rightarrow$ (III) by taking the divergence and curl of (I).

d) Show that (II) $\Leftrightarrow$ (III) using two boundary theorems.

Bonus: Derive a sixth formulation (VI) by calulating $\nabla^2\boldsymbol{E}$. Hint: take derivatives of the two equations in (III) to form the Laplacian and show that (III) $\Rightarrow$ (VI). This completes the symmetry of the differential vs. integral formulations: (I) $\Leftrightarrow$ (VI), (II) $\Leftrightarrow$ (III), and (IV) $\Leftrightarrow$ (V).

2. a) An amount of charge Q is uniformly distributed over a **spherical shell** of radius r'. Integrate Coulomb's law over this charge distribution to calculate the electric field $\boldsymbol{E}(r)$ everywhere (both inside and outside the shell). Hint: compare Griffiths, Example 2.7.

b) Compare part b) with the field of the same spherical charge distribution, this time calculated using Gauss' law.

c) Given a spherically symmetric charge distribution $\rho(r) = \rho_0(3 - 2r^2/r_0^2)e^{-r^2/r_0^2}$, where ρ_0 and r_0 are constants, plot $\rho(r)$ and show that the total charge everywhere is zero. Bonus: what is the total positive charge?

d) Explain why the charge in a shell of radius r' and thickness dr' is $đq = \rho(r')4\pi r'^2 dr'$. Calculate the electric field of the charge distribution in part c) by substituting $đq$ for Q in part a) and integrating over r'.

e) Calculate the divergence of the electric field $\boldsymbol{E}(r)$ of part d) and compare your answer with c).

f) Show that the impostor field $\boldsymbol{E}'(r) = \rho_0/\epsilon_0 \left[\hat{\boldsymbol{x}}(x-y) + \hat{\boldsymbol{y}}(x+y) + \hat{\boldsymbol{z}}z\right] e^{-r^2/r_0^2}$ has the same divergence, but calculate $\boldsymbol{\nabla}\times\boldsymbol{E}'$ to show that is cannot be the electric field for $\rho(r)$.

Also, Griffiths chapter 2, problems #4, 5, 6, 9, 10, 16, 18.