**University of Kentucky, Physics 416G**
**Problem Set #7, Rev. A, due Friday, 2014-11-07**

**1. Sturm-Liouville equation**. In class we saw that the Fourier series $\sin(kx)$ and $\cos(kx)$, and Legendre polynomials $P_\ell(\cos\theta)$ are all orthogonal. This holds in general for a large class of ODEs, including those from separation of variables of the Laplacian. We will prove this result using the same technique used for symmetric matrices (see the class notes on eigenvectors).

Let $y_1(x)$ and $y_2(x)$ be two solutions (*eigenfunctions*) of the differential equation $L[y_i] = \lambda_i\, y_i$ with real eigenvalues $\lambda_i$, where $L$ is a *linear differential operator* of the form

$$L[y(x)] \equiv \frac{1}{w(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] y(x) \tag{1}$$

acting on the function $y(x)$ with boundary conditions $y(a) = y(b) = 0$. We will show that these eigenfunctions are *orthogonal*,

$$\langle y_1|y_2\rangle \equiv \int_a^b w dx\, y_1 y_2 = 0 \qquad \text{if} \qquad \lambda_1 \neq \lambda_2. \tag{2}$$

Note that $L$ is just a shorthand for the above derivatives, but is analogous to a matrix operator, and the *inner product* $\langle y_1|y_2\rangle$ is also a shorthand for the above integral, which is analogous to the vector dot product. $y_1$, $y_2$, $p$, $q$, and $w$ are all functions of $x$, and $\lambda_1$ and $\lambda_2$ are real numbers.

**a)** Show by integration of parts that $\int_a^b y_1 \frac{d}{dx} y_2 dx = -\int_a^b y_2 \frac{d}{dx} y_1 dx$.

**b)** Extend part a) to show that $\int_a^b y_1 \frac{d}{dx} p \frac{d}{dx} y_2 dx = \int_a^b y_2 \frac{d}{dx} p \frac{d}{dx} y_1 dx$. Note that each derivative operator acts on everything to the right.

**c)** Use part b) to show that $\int_a^b w dx\, y_1 L[y_2] - \int_a^b w dx\, y_2 L[y_1] = 0$. We say that the operator $L$ is *self-adjoint* with respect to the weight $w dx$. It is the analog of a *symmetric* matrix.

**d)** Replace each operator in part c) with its eigenvalues, using $L[y_i] = \lambda_i\, y_i$ and factor out the inner product $\langle y_1|y_2\rangle$, to conclude that $y_i$ and $y_2$ are orthogonal if $\lambda_1 \neq \lambda_2$.

Bonus: give other examples of self-adjoint operators with orthogonal eigenfunctions, specifying the integration interval and weight function associated with each.

**2.** A **spherical capacitor** of radius $R$ is made of two hemispherical conductors separated by a negligible gap. The top hemisphere is held at potential $+V_0$ and the bottom hemisphere at $-V_0$. Evaluate the first two non-zero terms in each case.

**a)** Calculate the potential everywhere inside and outside the sphere. Note that both regions must be solved indpendently, because of the "external" boundary condition $V = \pm V_0$ at $r = R$.

**b)** Calculate the surface charge density on both the inside and outside of the sphere, and integrate the total charge on either hemisphere to determine the capacitance.

**c)** Draw the equipotentials and field lines, demonstrating the dominant features of the lowest non-zero multipole, both inside and outside the sphere.

Also, Griffiths 3ed[4ed] Ch. 3, #15[16], 23[24], 24[25].