University of Kentucky, Physics 416G
Problem Set #8, Rev. A, due Friday, 2014-11-14

1. We will use a spherical boundary value problem to develop a more systematic derivation of the **multipole expansion** than that presented in the text. Consider the potential $V(r,\theta)$ both inside and outside a spherical shell of radius r' due to a ring of total charge q on the shell at polar angle $\theta = \theta'$. In the limit $\theta' \to 0$, it is a point charge at the top of the sphere.

a) Write down a formula for $\sigma(\theta)$ of the ring charge using a delta function $\delta(\theta - \theta')$. Hint: make sure that $\int \sigma\, da = q$ if the region includes the ring charge, and 0 otherwise. In the same way, express the surface charge as a function of $x = \cos\theta$. Confirm the the change-of-variables formula for delta functions: $\delta(\theta - \theta')d\theta = \delta(x - x')dx$ where $x' = \cos\theta'$.

b) Solve for $V(r,\theta)$ both inside and outside the shell due to a uniform ring of charge q on the shell at the angle $\theta = \theta'$. Compare your answer for a point charge at the north pole ($\theta' = 0$) with $V(r) = q/4\pi\epsilon_0 \mathscr{r}$ to derive the addition formula of Griffiths Eq. 3.94.

c) Substitute $q \to \int đq'$ to obtain the general multipole expansion for an azimuthally symmetric charge distribution with multipoles $Q^{(\ell)}_{\text{ext,int}}$ discussed in class:

$$V_{\text{ext}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} Q^{(\ell)}_{\text{int}} \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \qquad \text{where} \qquad Q^{(\ell)}_{\text{int}} = \int đq'\, r'^{\ell} P_\ell(\cos\theta') \tag{1}$$

$$V_{\text{int}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} Q^{(\ell)}_{\text{ext}}\, r^{\ell} P_\ell(\cos\theta) \qquad \text{where} \qquad Q^{(\ell)}_{\text{ext}} = \int đq' \frac{P_\ell(\cos\theta')}{r'^{\ell+1}} \tag{2}$$

Note that the external multipole potential is only valid outside the entire charge distribution, while the internal multipole potential is only valid completely inside the charge distribution.

d) Identify monopole, dipole, and quadrupole terms, and compare with Griffiths. Sketch equipotentials and field lines of both the internal and external dipole. Match each multipole with the corresponding coefficient A_ℓ or B_ℓ in the general solution from separation of variables.

2. Calculate the (external) **quadrupole tensor** of two dipoles $\pm\boldsymbol{p}$ separated by a displacement $\boldsymbol{d}$ (pointing from $-\boldsymbol{p}$ to $+\boldsymbol{p}$). Consider two cases: a) $\boldsymbol{p} = p\hat{\boldsymbol{x}}$ and $\boldsymbol{d} = d\hat{\boldsymbol{y}}$; and b) $\boldsymbol{p} = p\hat{\boldsymbol{x}}$ and $\boldsymbol{d} = d\hat{\boldsymbol{x}}$. The other cases follow the same pattern.

Hint: construct each dipole $\boldsymbol{p}$ with two opposite charges separated appropriately, and calculate the quadrupole moment of all four point charges.

Bonus: a) calculate the general formula for $\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{d})$; b) compare and contrast the quadrupole tensor $\boldsymbol{Q}$ with the moment of inertia tensor $\boldsymbol{\mathcal{I}}$; c) compare with the hydrogen atom 3d orbitals.

Also, Griffiths 3ed[4ed] Ch. 3, #33[36], 38[44], 41[47], bonus: 45[52], 49[56].