University of Kentucky, Physics 416G
Problem Set #9, Rev. A, due Monday, 2014-11-24

1. The **Clausius-Mossotti** relation between atomic polarizability α ($\boldsymbol{p} = \alpha \boldsymbol{E}$ in a linear dielectric) and electric susceptibility χ_e ($\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$ in the same medium).

a) The simplest relation can be derived by ignoring the contribution from individual dipoles to $\boldsymbol{E}$. Using the definition of $\boldsymbol{P}$, explain why $\boldsymbol{P} = N\boldsymbol{p}$, where N is the number density of molecules (number/volume). Show that in this case $N\alpha = \epsilon_0 \chi_e$.

b) Correct for the fact a dipole does not feel its own field. Thus the definition of polarizability in a dielectric material must be modified to $\boldsymbol{p} = \alpha \boldsymbol{E}_0$, where $\boldsymbol{E}_0$ is the macroscopic field due to everything but $\boldsymbol{p}$. Calculate $\boldsymbol{E}_0$ at the location of $\boldsymbol{p}$ as a function of the macroscopic field $\boldsymbol{E}$ and susceptibility χ_e assuming that $\boldsymbol{p}$ is at the center of a bubble of radius R (the size of the molecule) inside the dielectric material. Hint: see example 4.2 and problem 4.16 [both editions].

c) Using the modified equation from part a), $N\alpha \boldsymbol{E}_0 = \epsilon_0 \chi_e \boldsymbol{E}$, and part b), derive the relation $N\alpha = \epsilon_0 \chi_e \frac{3}{3+\chi_e} = 3\epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 2}$. Note this reduces to the naïve relation as $\chi_e \to 0$.

d) Use part a) to compare the α and χ_e for elements which appear in both tables 4.1 and 4.2. Calculate the fractional correction term $\frac{3}{3+\chi_e}$ derived in part c).

2. The **Langevin** formula. In this problem we calculate the electric susceptibility χ_e of a polar material, composed of molecules with permanent dipole moment $\boldsymbol{p}$. While the dipole has a tendency to align along the total field $\boldsymbol{E}$, thermal agitations prevent perfect alignment. From statistical mechanics, the number of atoms in state i of energy U_i is proportional to the Boltzman factor $w = e^{-U_i/kT}$, where k is the Boltzman constant and T the temperature of the system. Thus colder systems are more likely to settle into the lowest energy state. In our case, the state of a dipole in an electric field is given by its rotational direction $i \equiv (\theta, \phi)$ with respect to the field $\boldsymbol{E} = E\hat{\boldsymbol{z}}$.

a) Calculate the potential energy $U(\theta)$ of the dipole $\boldsymbol{p}$ in the field $\boldsymbol{E}$.

b) By considering the solid angle of a ring of constant energy (ie. constant θ), show that the number of states dn of energy between $U(\theta)$ and $U(\theta + d\theta)$ is proportional to $\sin\theta d\theta$. The ratio $g = \frac{dn}{d\theta}$ is called the *degeneracy of states*, the *density of states*, or the *phase space*.

c) Calculate the average energy $\langle U \rangle \equiv \int dn\, w(\theta) U(\theta) / \int dn\, w(\theta)$ of a dipole p in the field E, weighted by the Boltzman distribution $w(\theta)$.

d) Calculate the weighted average of dipole moment along the field $\langle p_z \rangle$ of $\langle p \cos\theta \rangle$ as a function of the electric field E and temperature T, also weighted by the Boltzman distribution. Explain why $P = N\langle p_z \rangle$ and show that $P = Np[\coth(pE/kT) - (kT/pE)]$. Graph this as a function of E.

e) In the limit where $pE \ll kT$, determine the linear relationship between E and P to determine the susceptibility χ_e. Calculate χ_e for liquid and vapor water, using $p = 3.8 \times 10^{-9}\ e\,\mathrm{cm}$.

Also, Griffiths 3ed[4ed] Ch. 4, #4[4], 5[5], 6[6], 7[7], 8[8], 10[10], 14[14].