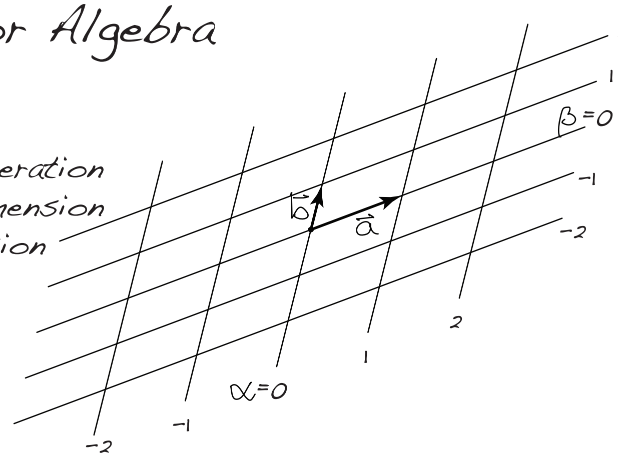
# Section 1.1 - Vector Algebra




* Linear spaces

~ linear combination: $(\alpha\vec{u}+\beta\vec{v})$ is the basic operation

~ basis: $(\hat{x},\hat{y},\hat{z}$ or $\vec{a},\vec{b},\vec{c})$ # basis elements = dimension

independence: not collapsed into lower dimension

closure: vectors span the entire space

~ components: $\vec{X} = \vec{a}\alpha + \vec{b}\beta + \vec{c}\gamma = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$

in matrix form: $\vec{X} = \mathbb{b}\,\mathbb{X}$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

where

$$\vec{a} = \hat{x}a_x + \hat{y}a_y + \hat{z}a_z = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}$$

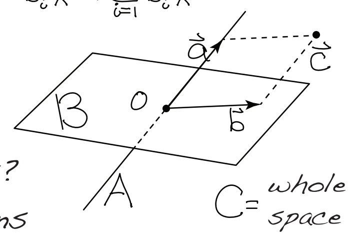
(usually one upper, one lower index)

~ Einstein notation: implicit summation over repeated indices $\vec{x} = \vec{b}_i x^i \equiv \sum_{i=1}^{3}\vec{b}_i x^i$

~ direct sum: $C = A \oplus B$ add one vector from each independent space to get vector in the product space (not simply union)

~ projection: the vector $\vec{c}=\vec{a}+\vec{b}$ has a unique decomposition ('coordinates' $(\vec{a},\vec{b})$ in $A, B$) - relation to basis/components?

~ all other structure is added on as multilinear (tensor) extensions



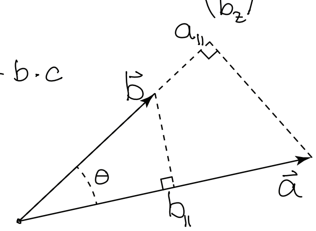

* Metric (inner, dot) product - distance and angle

$$c = \vec{a}\cdot\vec{b} = ab\cos\theta = a_{\parallel}b = ab_{\parallel} = a_xb_x + a_yb_y + a_zb_z = a_ib^i = (a_x a_y a_z)\begin{pmatrix}b_x\\ b_y\\ b_z\end{pmatrix}$$

~ properties: 1) scalar valued - what is outer product?

2) bilinear form $a\cdot(b+c) = a\cdot b + a\cdot c$ $\quad (a+b)\cdot c = a\cdot c + b\cdot c$

3) symmetric $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$




~ orthonormality and completeness - two fundamental identities help to calculate components, implicitly in above formulas

$$\boxed{\begin{aligned}\hat{e}_i\cdot\hat{e}_j &= \delta_{ij}\\ \sum_{i=1}^{3}\hat{e}_i\hat{e}_i\cdot &= I\end{aligned}}$$

Kroneker delta: components of the identity matrix $\delta_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases} \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I$

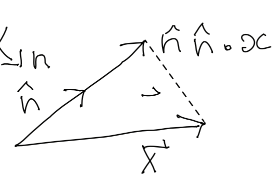
$\vec{x} = \vec{b}_i x^i \equiv \quad \vec{b}_i x^i$

$$a^i = \vec{a}\cdot\hat{e}^i = a^1\hat{e}_1\cdot\hat{e}^i + a^2\hat{e}_2\cdot\hat{e}^i + a^3\hat{e}_3\cdot\hat{e}^i$$

~ orthogonal projection: a vector $\hat{n}$ divides the space $X$ into $X_{\parallel n}\oplus X_{\perp n}$

geometric view: dot product $\hat{n}\cdot\vec{x}$ is length of $\vec{x}$ along $\hat{n}$

Projection operator: $P_{\parallel} \equiv \hat{n}\hat{n}\cdot$ acts on $x$: $P_{\parallel}\vec{x} = \vec{x}_{\parallel} = \hat{n}\hat{n}\cdot\vec{x}$



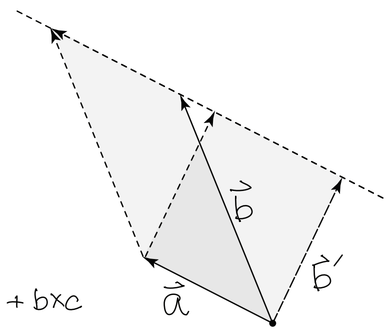

~ generalized metric: for basis vectors which are not orthonormal, collect all nxn dot products into a symmetric matrix (metric tensor) $g = \begin{pmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}$

$$g_{ij} = \vec{b}_i\cdot\vec{b}_j \qquad \vec{x}\cdot\vec{y} = x^i\vec{b}_i\cdot\vec{b}_j x^j = x^i g_{ij} x^j = (x^1 x^2 x^3)\begin{pmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}\begin{pmatrix}y^1\\ y^2\\ y^3\end{pmatrix}$$

$$= \mathbb{X}^T\mathbb{b}^T\cdot\mathbb{b}\mathbb{X} = \mathbb{X}^T g\,\mathbb{X} =$$

in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)

# Exterior Products – higher-dimensional objects




$\vec a \times \vec b = \vec a \times \vec b'$
$\vec a \times (\vec b - \vec b') = \vec 0$
(parallel)

* cross product (area)

$$\vec c = \vec a \times \vec b = \hat n \, a b \sin\theta = \hat n \, a_\perp b = \hat n \, a b_\perp = \begin{vmatrix} \hat x & \hat y & \hat z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat n \perp \vec a$ and $\hat n \perp \vec b$ (RH-rule)

~ properties: 1) vector-valued
2) bilinear $\quad a \times (b + c) = a \times b + a \times c \quad (a + b) \times c = a \times c + b \times c$
3) antisymmetric $\quad \vec a \times \vec b = -\vec b \times \vec a$

~ components: $\hat e_i \times \hat e_j = \varepsilon_{ij}{}^k \hat e_k \qquad \varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ even permutation} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index} \end{cases}$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

Levi-Civita tensor – completely antisymmetric:

$$\vec x \times \vec y = x^i \vec b_i \times \vec b_j y^j = \varepsilon_{ij}{}^k x^i y^j \hat e_k$$

~ orthogonal projection: $\hat n \times$ projects $\perp$ to $\hat n$ and rotates by $90°$

$$\hat x_\perp = -\hat n \times (\hat n \times \hat x) = P_\perp \vec x \qquad P_\perp = -\hat n \times \hat n \times \qquad \boxed{P_\parallel + P_\perp = \hat n \hat n \cdot - \hat n \times \hat n \times = I}$$

~ where is the metric in $\times$?
vector × vector = pseudovector
symmetries act more like a 'bivector'
can be defined without metric

* triple product (volume of parallelpiped) – base times height $\quad d = \vec a \cdot \vec b \times \vec c = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$
~ completely antisymmetric – definition of determinant
~ why is the scalar product symmetric / vector product antisymmetric?
~ vector vector × vector = pseudoscalar (transformation properties)
~ acts more like a 'trivector' (volume element)
~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)
~ extended vector space with basis elements from objects of each dimension
~ pseudo-vectors, scalar separated from normal vectors, scalar

| magnitude, | length, | area, | volume |
|---|---|---|---|
| scalar, | vectors, | bivectors, | trivector |
| $1$, | $\hat x, \hat y, \hat z$, | $\hat x \hat y, \hat y \hat z, \hat z \hat x$, | $\hat x \hat y \hat z$ |

~ what about higher-dimensional spaces (like space-time)?
can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements
most important example: BAC-CAB rule (HW1: relation to projectors)

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$\varepsilon^i{}_{jk} A^j (\varepsilon^k{}_{mn} B^m C^n) = (\delta^i_m \delta_{jn} - \delta^i_n \delta_{jm}) A^j B^m C^n = B^i (A^j C_j) - C^i (A^j B_j)$$