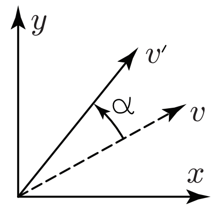
Section 1.1.5 - Linear Operators

* Linear Transformation
  - ~ function which preserves linear combinations
  - ~ determined by action on basis vectors (egg-crate)
  - ~ rows of matrix are the image of basis vectors
  - ~ determinant = expansion volume (triple product)
  - ~ multilinear (2 sets of bases) - a tensor

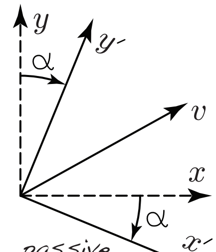
$$M(\alpha\vec{a}+\beta\vec{b}) = \alpha M(\vec{a}) + \beta M(\vec{b})$$

$$M\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{M\begin{pmatrix}1\\0\end{pmatrix}}_{\vec{m}_1} x + \underbrace{M\begin{pmatrix}0\\1\end{pmatrix}}_{\vec{m}_2} y = \left(\begin{array}{c|c} m_{1x} & m_{2x} \\ m_{1y} & m_{2y}\end{array}\right)\begin{pmatrix}x\\y\end{pmatrix}$$

* Change of coordinates
  - ~ two ways of thinking about transformations both yield the same transformed components
  - ~ active: basis fixed, physically rotate vector
  - ~ passive: vector fixed, physically rotate basis



active transformation



passive transformation

* Transformation matrix (active) - basis vs. components

$$(\vec{a}\,\vec{b}\,\vec{c}) = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z\end{pmatrix} \qquad \vec{X} = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

$$\vec{\mathbb{e}}' = \vec{\mathbb{e}}\,\mathcal{R} \qquad \vec{x} = \overbrace{\vec{\mathbb{e}}'}\,\mathbb{X}' = \overbrace{\vec{\mathbb{e}}\,\mathcal{R}}\underbrace{\mathcal{R}\,\mathbb{X}'} = \vec{\mathbb{e}}\,\underbrace{\mathbb{X}} = \vec{x} \qquad \mathbb{X} = \mathcal{R}\;\mathbb{X}'$$

$$\boxed{\begin{array}{c}\vec{\mathbb{e}}' = \vec{\mathbb{e}}\,\mathcal{R} \\ \mathbb{X}' = \mathcal{R}^{-1}\mathbb{X}\end{array}}$$

* Orthogonal transformations
  - ~ R is orthogonal if it 'preserves the metric' (has the same form before and after)

$$\vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}} = \begin{pmatrix}\hat{x}\\\hat{y}\end{pmatrix}\cdot(\hat{x}\;\hat{y}) = \begin{pmatrix}\hat{x}\cdot\hat{x} & \hat{x}\cdot\hat{y} \\ \hat{y}\cdot\hat{x} & \hat{y}\cdot\hat{y}\end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} = g \qquad \vec{\mathbb{e}}'^T\cdot\vec{\mathbb{e}}' = \begin{pmatrix}\vec{a}\\\vec{b}\end{pmatrix}\cdot(\vec{a}\;\vec{b}) = \begin{pmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{pmatrix} = g'$$

$$\vec{\mathbb{e}}' = \vec{\mathbb{e}}\,\mathcal{R} \qquad \vec{\mathbb{e}}'^T\cdot\vec{\mathbb{e}}' = \mathcal{R}^T\vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}}\,\mathcal{R} = \mathcal{R}^T g\,\mathcal{R} = g' \qquad g = g' \qquad \boxed{\mathcal{R}^T g\,\mathcal{R} = g}$$

  - ~ equivlent definition in terms of components:

$$\vec{X}\cdot\vec{X} = \overbrace{\mathbb{X}^T}\,\overbrace{g}\,\mathbb{X} = \overbrace{\mathbb{X}'^T\mathcal{R}^T}\,\underbrace{g}\,\overbrace{\mathcal{R}\,\mathbb{X}'} = \mathbb{X}'^T g'\,\mathbb{X}' \qquad \text{(metric invariant under rotations if } g = g'\text{)}$$

  - ~ starting with an orthonormal basis: $\quad g = I \quad g_{ij} = \delta_{ij} \qquad \mathcal{R}^T\mathcal{R} = I \qquad \mathcal{R}^{-1} = \mathcal{R}^T$

* Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition
  - ~ recall complex numbers $\quad u = \rho + i\phi \qquad \rho^* = \rho \qquad (i\phi)^* = -i\phi$

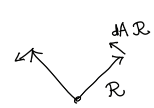
$$e^u = e^{\rho+i\phi} = r e^{i\phi} \qquad |e^{i\phi}|^2 = e^{-i\phi}e^{i\phi} = e^{i0} = 1$$

  - ~ similar behaviour of symmetric / antisymmetric matrices

M artibrary matrix
T symmetric
A antisymmetric

S symmetric
R orthogonal

$$M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d\end{pmatrix} + \begin{pmatrix} 0 & (b-c)/2 \\ (c-b)/2 & 0\end{pmatrix} = T + A$$



$$e^M = 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots = e^{T+A} \neq e^T e^A$$

$$S = e^T = e^{VWV^{-1}} = V e^W V^{-1} \qquad R = e^A \qquad R^T R = (e^A)^T e^A = e^{A^T+A} = e^0 = I$$

$$\det\begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots\end{pmatrix} = e^{\lambda_1}\cdot e^{\lambda_2}\cdots = e^{\lambda_1+\lambda_2+\dots} = e^{\mathrm{tr}\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}} \qquad \det e^A = e^{\mathrm{tr}\,A} = e^0 = 1$$

Eigenparaphernalia

* illustration of symmetric matrix S with eigenvectors v, eigenvalues λ

$$S\,v = \lambda v$$

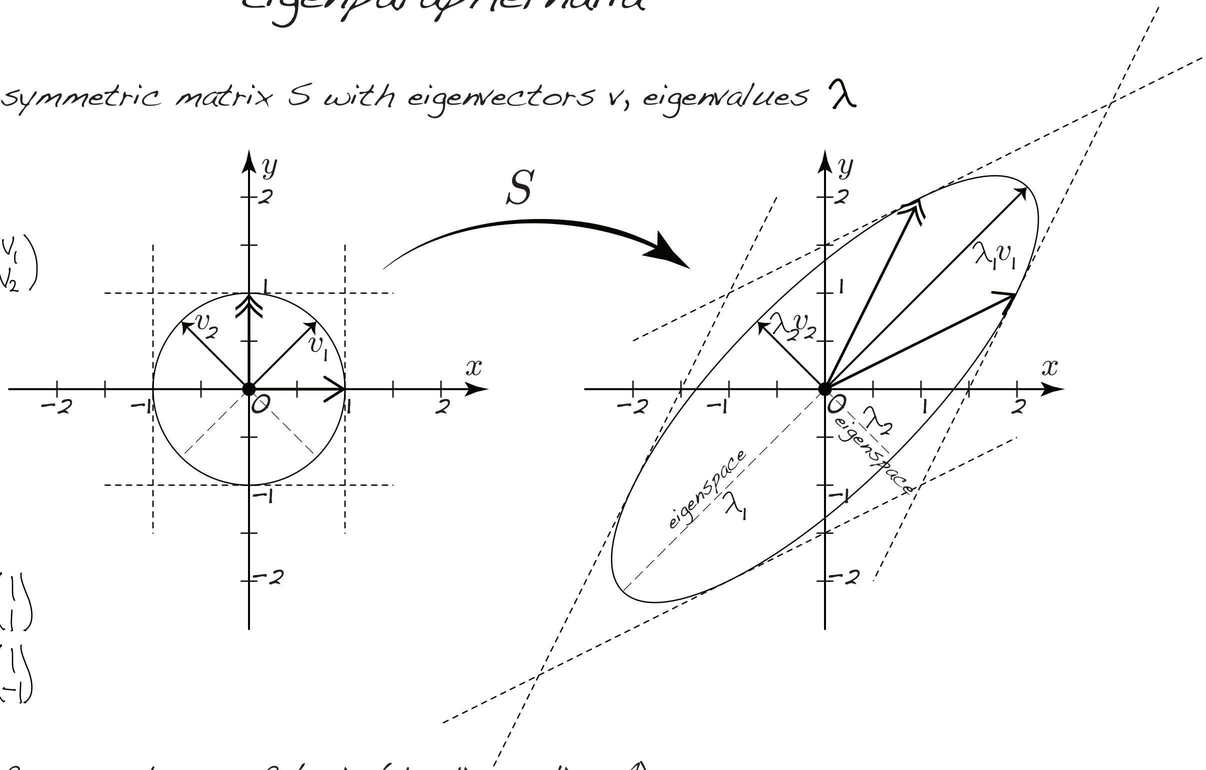
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



* similarity transform – change of basis (to diagonalize A)

$$S\,(v_1 v_2 \ldots) = (\vec{v}_1 \vec{v}_2 \ldots)\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \qquad S\,V = V\,W\,V^{-1} = V\,W\,V^T$$

* a symmetric matrix has real eigenvalues

$$S\,v = \lambda v \qquad v^{*T} S\,v = \lambda\, v^{*T} v$$

$$v^{*T} S = v^{*T} \lambda^* \qquad v^{*T} S\,v = \lambda^*\, v^{*T} v \qquad \lambda = \lambda^*$$

~ what about a antisymmetric/orthogonal matrix?

* eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal

$$v^T S = (S^T v)^T = (S v)^T = (\lambda v)^T = v^T \lambda$$

$$\lambda_1\, v_1 \cdot v_2 = (v_1^T S)\, v_2 = v_1^T (S\, v_2) = v_1 \cdot v_2\, \lambda$$

$$v_1 \cdot v_2\, (\lambda_1 - \lambda_2) = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \text{ then } v_1 \cdot v_2 = 0.$$

* singular value decomposition (SVD)

~ transformation from one orthogonal basis to another

$$M = R\,S = \underbrace{R\,V}\,W\,V^T = U\,W\,V^T$$

~ extremely useful in numerical routines

M arbitrary matrix
R orthogonal
S symmetric
W diagonal matrix
V orthogonal (domain)
U orthogonal (range)