Section 1.2 – Differential Calculus

* differential operator

~ ex. $u = x^2 \quad du = dx^2 = 2x\,dx$ $\qquad d \equiv \lim_{\Delta \to 0} \Delta \approx 0$

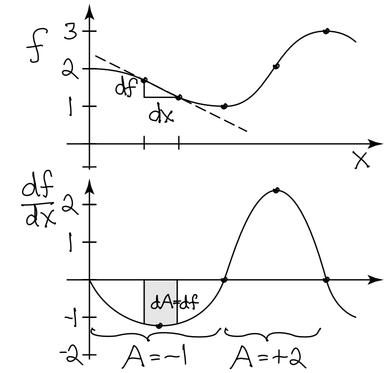
or $d(\sin x^2) = \cos(x^2)\,d\,x^2 = \cos x^2 \cdot 2x \cdot dx$

~ df and dx connected – refer to the same two endpoints

~ made finite by taking ratios (derivative or chain rule)

or infinite sum = integral (Fundamental Thereom of calculus)

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \int_a^b \frac{df}{dx}\,dx = \int_a^b df = f\Big|_a^b$$

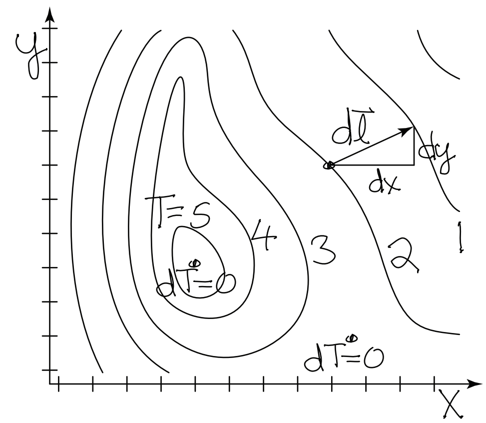


* scalar and vector fields – functions of position ($\vec{r}$)

~ "field of corn" has a corn stalk at each point in the field

~ scalar fields represented by level curves (2d) or surfaces (3d)

~ vector fields represented by arrows, field lines, or equipotentials



* partial derivative & chain rule

~ signifies one varying variable AND other fixed variables

~ notation determined by denominator; numerator along for the ride

~ total variation split into sum of variations in each direction

$$\frac{\partial u}{\partial x} \quad \left(\frac{\partial u}{\partial x}\right)_{y,z} \quad \partial_x u \quad u_{,x} \qquad \frac{\cdots}{\cdots} = \frac{dx}{\cdots}\frac{\cdots}{\partial x} + \frac{dy}{\cdots}\frac{\cdots}{\partial y} + \frac{dz}{\cdots}\frac{\cdots}{\partial z}$$

* vector differential – gradient

~ differential operator , del operator

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$
$$= \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla} T \cdot \underbrace{(dx, dy, dz)}_{d\vec{\ell}}$$

$$\boxed{\begin{aligned} d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{r}\cdot\nabla \\ \nabla &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \frac{d}{d\vec{r}} \\ d\vec{\ell} &= \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz = d\vec{r} \end{aligned}}$$

~ differential line element: $\cdot d\vec{\ell}$ and $\frac{}{d\vec{\ell}}$ transforms between $\hat{x}, \hat{y}, \hat{z} \leftrightarrow dx, dy, dz$ and $d \leftrightarrow \nabla$

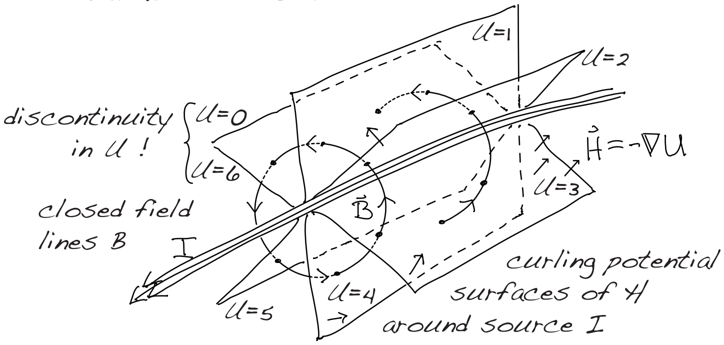
~ example: $d\,x^2 y = 2xy\,dx + x^2 dy = (2xy, x^2)\cdot(dx, dy)$

~ example: let $z = f(x,y)$ be the graph of a surface. What direction does ∇f point?

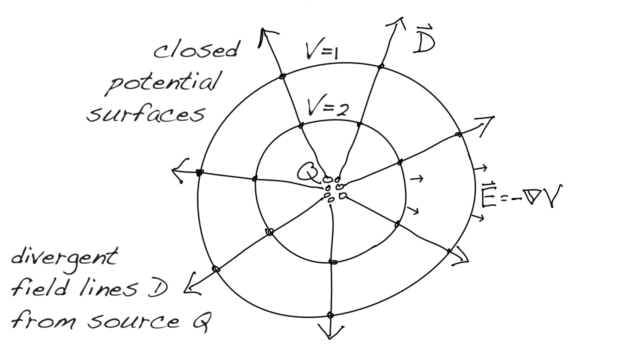
now let $g = z - f(x,y)$ so that $g = 0$ on the surface of the graph

then $\nabla g = (-f_{,x}, -f_{,y}, 1)$ is normal to the surface

* illustration of curl



* illustration of divergence



Higher Dimensional Derivatives

* curl - circular flow of a vector field

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix} = \begin{matrix} \hat{x}\,(V_{z,y}-V_{y,z}) \\ +\hat{y}\,(V_{x,z}-V_{z,x}) \\ +\hat{z}\,(V_{y,x}-V_{x,y}) \end{matrix}$$

* divergence - radial flow of a vector field

$$\nabla\cdot\vec{V} = (\partial_x\,\partial_y\,\partial_z)\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = V_{x,x}+V_{y,y}+V_{z,z}$$

* product rules
  - ~ how many are there?
  - ~ examples of proofs

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$\vec{A}\times(\nabla\times\vec{B}) = \nabla(\vec{A}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\nabla)$$

$$\nabla\times(\vec{A}\times B) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

$$\nabla(fg) = \nabla f\cdot g + f\cdot\nabla g$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\times(f\vec{A}) = \nabla f\times\vec{A} + f(\nabla\times\vec{A})$$

$$\nabla\times(\vec{A}\times\vec{B}) = (B\cdot\nabla)A - B(\nabla\cdot A) - (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\cdot(f\vec{A}) = \nabla f\cdot\vec{A} + f\,\nabla\cdot\vec{A}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})$$

* second derivatives - there is really only ONE! (the Laplacian) $\boxed{\nabla^2 \equiv \nabla\cdot\nabla \equiv \partial_x^2+\partial_y^2+\partial_z^2}$

1) $\nabla\cdot(\nabla T) = \nabla^2 T$ ~ eg: $\nabla^2 T = 0$ no net curvature - stretched elastic band

$(\nabla\cdot\nabla)\vec{v} = \nabla^2\vec{v}$ ~ defined component-wise on v_x, v_y, v_z (only cartesian coords)

3), 5) $\nabla^2 = \nabla_\parallel^2 + \nabla_\perp^2$ ~ longitudinal / transverse projections $\nabla(\nabla\cdot\vec{v}) \equiv \nabla_\parallel^2\vec{v}$

$= \nabla(\nabla\cdot\ \ -\nabla\times\nabla\times$ $\qquad \vec{k}\cdot\vec{k} = \vec{k}\vec{k}\cdot\ - \vec{k}\times(\vec{k}\times$ $\qquad -\nabla\times\nabla\times\vec{v} \equiv -\nabla_\perp^2\vec{v}$

2), 4) $\nabla\times\nabla = 0$ $\qquad \nabla\times(\nabla T) = 0 \qquad \nabla\cdot(\nabla\times\vec{v}) = 0$

$$\nabla\times\nabla = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = \begin{matrix} \hat{x}\,(\partial_y\partial_z-\partial_z\partial_y) \\ +\hat{y}\,(\partial_z\partial_x-\partial_x\partial_z) \\ +\hat{z}\,(\partial_x\partial_y-\partial_y\partial_x) \end{matrix}$$

~ equality of mixed partials ($d^2=0$)

* unified approach to all higher-order derivatives with differential operator

1) $d^2 = 0$ 2) $dx^2 = 0$ 3) $dx\,dy = -dy\,dx$ + differential (line, area, volume) elements

~ Gradient

$$df = f_{,x}\,dx + f_{,y}\,dy + f_{,z}\,dz = \nabla f\cdot d\vec{l} \qquad d\vec{l} = (dx, dy, dz) = d\vec{r}$$

~ Curl

$$d(\vec{A}\cdot d\vec{l}) =_{g} d(A_x dx + A_y dy + A_z dz)$$
$$= A_{x,x}\,\cancel{dxdx} + A_{x,y}\,dydx + A_{x,z}\,dzdx$$
$$+ A_{y,x}\,dxdy + A_{y,y}\,\cancel{dydy} + A_{y,z}\,dzdy$$
$$+ A_{z,x}\,dxdz + A_{z,y}\,dydz + A_{z,z}\,\cancel{dzdz}$$
$$= (A_{z,y}-A_{y,z})dydz + (A_{x,z}-A_{z,x})dzdx + (A_{y,x}-A_{x,y})dxdy$$

$$d(\vec{A}\cdot d\vec{l}) = (\nabla\times\vec{A})\cdot d\vec{a}$$

$$d\vec{a} = (dydz, dzdx, dxdy) = \tfrac{1}{2}d\vec{l}\times d\vec{l} = d^2\vec{r}$$

~ Divergence

$$d(\vec{B}\cdot d\vec{a}) = d(B_x\,dydz + B_y\,dzdx + B_z\,dxdy)$$
$$= B_{x,x}\,dxdydz + B_{x,y}\,\cancel{dydydz} + B_{x,z}\,\cancel{dzdydz}$$
$$+ B_{y,x}\,\cancel{dxdzdx} + B_{y,y}\,dydzdx + B_{y,z}\,\cancel{dzdzdx}$$
$$+ B_{z,x}\,\cancel{dxdxdy} + B_{z,y}\,\cancel{dydxdy} + B_{z,z}\,dzdxdy.$$
$$= (B_{x,x}+B_{y,y}+B_{z,z})\,dxdydz$$

$$d(\vec{B}\cdot d\vec{a}) = \nabla\cdot B\cdot d\tau \qquad d\tau = \tfrac{1}{6}d\vec{l}\cdot d\vec{l}\times d\vec{l} = d^3\vec{r}$$

$$\boxed{\nabla f = \frac{df}{d\vec{l}} = \frac{df}{d\vec{r}}} \qquad \boxed{\nabla\times\vec{A} = \frac{d(\vec{A}\cdot d\vec{l})}{d\vec{a}} = \frac{d(d\vec{r}\cdot\vec{A})}{d^2\vec{r}}} \qquad \boxed{\nabla\cdot\vec{B} = \frac{d(\vec{B}\cdot d\vec{a})}{d\tau} = \frac{d(d^2\vec{r}\cdot\vec{B})}{d^3\vec{r}}}$$