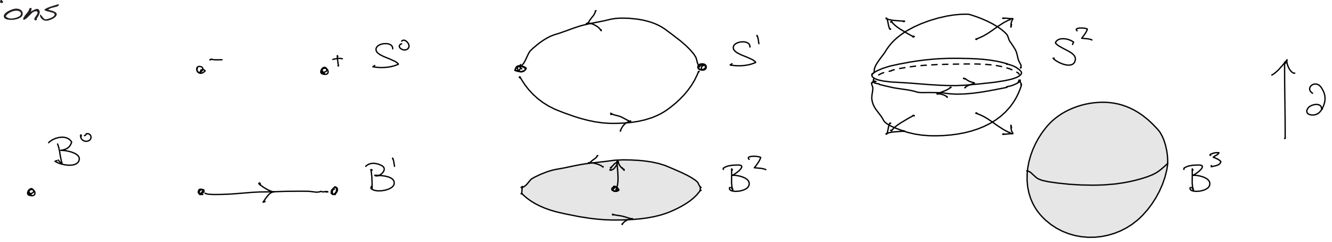
Section 1.3.2-5 – Region | Form = Integral

* Regions



~ definition of boundary operator '∂'
'closed' region (cycle): $\partial S = 0$

~ a boundary is always closed $\boxed{\partial\partial R = 0}$

~ is every closed closed region a boundary?
$$\partial S = 0 \iff S = \partial R$$

~ a room (walls, window, ceiling, floor)
is CLOSED if all doors, windows closed
is OPEN if the door or window is open;

~ what is the boundary?

~ think of a surface that has loops
that do NOT wrap around disks!

* Forms – see last notes

~ combinations of scalar/vector fields and differentials so they can be integrated

~ pictoral representation enables 'integration by eye'

RANK	NOTATION	REGION	VISUAL REP.	DERIVATIVE
scalar	$\omega^{(0)} = f$	Q point	level surfaces	$d\omega^{(0)} = \nabla f \cdot d\vec{l}$
vector	$\omega^{(1)} = \tilde{A} = \vec{A}\cdot d\vec{l}$	P path	equipotentials	$d\omega^{(1)} = \nabla\times\vec{A}\cdot d\vec{a}$
p-vector	$\omega^{(2)} = \tilde{B} = \vec{B}\cdot d\vec{a}$	S surface	flux tubes	$d\omega^{(2)} = \nabla\cdot B\, d\tau$
p-scalar	$\omega^{(3)} = \tilde{\rho} = \rho\, d\tau$	V volume	subst boxes	$d\omega^{(3)} = 0$

∂ ∂ ∂ ∂

d d d d

edge of the world!

~ properties of differential operator 'd'
transforms form into higher-dimensional form, sitting on the boundary

~ Poincare lemma $\boxed{dd\omega = 0}$ $\qquad \nabla\times\nabla V = 0 \qquad \nabla\cdot\nabla\times\vec{A} = 0$

~ converse – existance of potentials $V, \vec{A}$
$$d\omega = 0 \iff \omega = d\alpha \qquad \nabla\times E = 0 \iff E = -\nabla V \qquad \nabla\cdot\vec{B} = 0 \iff \vec{B} = \nabla\times\vec{A}$$
for space without any n-dim 'holes' in it

* Integrals – the overlap of a region on a form = integral of form over region

~ regions and forms are dual – they combine to form a scalar

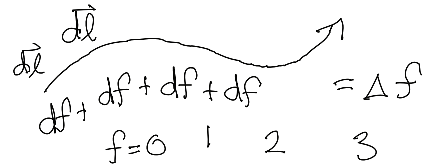
~ generalized Stoke's therem:
'∂' and 'd' are adjoint operators – they have the same effect in the integral

$$\int_R d\omega = \oint_{\partial R} \omega \qquad \text{note: } 0 = \int_{\partial\partial R} \omega = \int_{\partial R} d\omega = \int_R dd\omega = 0$$

Generalized Stokes Theorem

* Fundamental Theorem of Vector Calculus: 0d–1d

$$\int_a^b \nabla\varphi\cdot d\vec{l} = \int_a^b df = f(b)-f(a)$$



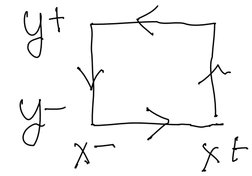
* Stokes' Thereom: 1d–2d

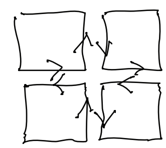
$$\nabla\times\vec{A}\cdot d\vec{a} = \frac{\partial A_y}{\partial x}dx\,dy - \frac{\partial A_x}{\partial y}dx\,dy + \ldots$$

$$= A_y(x^+)dy + A_y(x^-)(-dy) + A_x(y^+)(-dx) + A_x(y^-)dx + \ldots$$

$$= \Sigma\,\vec{A}\cdot d\vec{l} \text{ around boundary}$$

$$+ \text{ other faces}$$



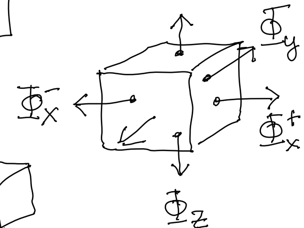


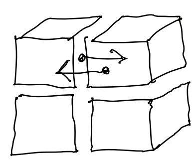
* Stokes' Thereom: 2d–3d

$$\nabla\cdot\vec{B}\,d\tau = \frac{\partial B_x}{\partial x}dx\,dy\,dz + \frac{\partial B_y}{\partial y}dy\,dz\,dx + \frac{\partial B_z}{\partial z}dz\,dx\,dy$$

$$= B_x(x^+)dy\,dz + B_x(x^-)(-dy\,dz) + 4 \text{ other faces}$$

$$= \Sigma\,\vec{B}\cdot d\vec{a} \text{ around boundary}$$





* note: all interior $f(x)$, flow, and flux cancel at opposite edges
* proof of converse Poincare lemma: integrate form out to boundary
* proof of gen. Stokes theorem: integrate derivative out to the boundary

$$\int_R d\omega = \oint_{\partial R}\omega \iff \int_P \nabla\varphi\cdot d\vec{l} = \oint_{\partial P}\varphi \qquad \int_S \nabla\times\vec{A}\cdot d\vec{a} = \oint_{\partial S}\vec{A}\cdot d\vec{l} \qquad \int_R \nabla\cdot\vec{B}\,d\tau = \oint_{\partial R}\vec{B}\cdot d\vec{a}$$

* example – integration by parts

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}f\right) = \left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f + \frac{\hat{r}}{r^2}\cdot\nabla f$$

$$\int_V \frac{\hat{r}}{r^2}\cdot\nabla f\,d\tau = \int_V \nabla\cdot\left(\frac{\hat{r}}{r^2}f\right)\cdot d\tau - \int_V\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f\,d\tau$$

$$\int_V \frac{1}{r^2}\frac{\partial f}{\partial r}r^2dr\cdot d\Omega = \oint_{\partial V} d\vec{a}\cdot\frac{\hat{r}}{r^2}f - \int_V 4\pi\,\delta^3(\vec{r})\,f\,d\tau$$

$$\int d\Omega\int_{r=0}^{R} df = \int r^2 d\Omega\,\hat{r}\cdot\frac{\hat{r}}{r^2}f - 4\pi f(0)$$

$$\int d\Omega\, f(R) - f(0) = \int d\Omega\, f(R,\theta,\phi) - 4\pi f(0)$$

$$4\pi\left[\langle f\rangle_R - f(0)\right] = 4\pi\left[\langle f\rangle_R - f(0)\right]$$