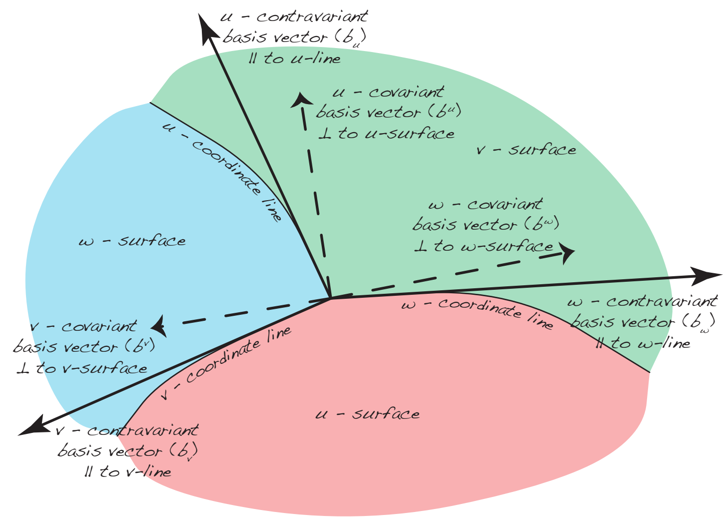
Curvilinear Coordinates

* coordinate surfaces and lines

~ each coordinate is a scalar field $q(\vec{r})$

~ coordinate surfaces: constant q^i

~ coordinate lines: constant q^j, q^k



* coordinate basis vectors

$q^i \sim \{u, v, w\}$ ~ generalized coordinates

$\vec{b}_i = \left(\frac{\partial \vec{r}}{\partial q^i}\right)_{q^j, q^k} \sim \{\hat{u}f, \hat{v}g, \hat{w}h\}$ ~ contravariant basis

$\vec{b}^i = \nabla q^i \sim \{\hat{u}/f, \hat{v}/g, \hat{w}/h\}$ ~ covariant basis

$h_i = |\vec{b}_i| \sim \{f, g, h\}$ ~ scale factor

$\hat{e}_i = \vec{b}_i / h_i \sim \{\hat{u}, \hat{v}, \hat{w}\}$ ~ unit vector

$g_{ij} = \vec{b}_i \cdot \vec{b}_j \sim \begin{pmatrix} h_1 & 0 & 0^2 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$ ~ metric (dot product)

$\vec{\Gamma}_{ij} = \frac{\partial \vec{b}_j}{\partial q^i} = \vec{b}_k \Gamma^k_{ij}$ ~ Christoffel symbols - derivative of basis vectors

* differential elements

$$d\vec{l} = \frac{\partial \vec{r}}{\partial q^1} dq^1 + \frac{\partial \vec{r}}{\partial q^2} dq^2 + \frac{\partial \vec{r}}{\partial q^3} dq^3 = \vec{b}_i \, dq^i$$

$$= \hat{e}_1 \underbrace{h_1 dq^1}_{dl_1} + \hat{e}_2 \underbrace{h_2 dq^2}_{dl_2} + \hat{e}_3 \underbrace{h_3 dq^3}_{dl_3}$$

$$d\vec{a} = \tfrac{1}{2} d\vec{l} \times d\vec{l} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \end{vmatrix}$$

$$= \hat{e}_1 \cdot h_2 dq^2 \cdot h_3 dq^3 + \hat{e}_2 \cdot h_3 dq^3 \cdot h_1 dq^1 + \hat{e}_3 \cdot h_1 dq^1 \cdot h_2 dq^2$$

$$d\tau = \tfrac{1}{2} d\vec{l} \times d\vec{a} = \tfrac{1}{2} d\vec{l} \cdot d\vec{l} \times d\vec{l} = h_1 dq^1 \cdot h_2 dq^2 \cdot h_3 dq^3$$

* example

$(c_\phi \equiv \cos\phi)$

$x = s \qquad dx = c_\phi \, ds - s \, s_\phi \, d\phi$

$y = s \, s_\phi \qquad dy = s_\phi \, ds + s \, c_\phi \, d\phi$

$$d\vec{l} = \hat{x} dx + \hat{y} dy = (\hat{x} c_\phi + \hat{y} s_\phi) ds + (\hat{x} s_\phi - \hat{y} c_\phi) s \, d\phi$$

$$= \hat{s} \, ds + \hat{\phi} \, s \, d\phi \qquad (\hat{s} \;\; \hat{\phi}) = (\hat{x} \;\; \hat{y}) \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$$

$s^2 = x^2 + y^2 \qquad 2s\,ds = 2x\,dx + 2y\,dy$

$y = x \tan\phi \qquad dy = dx \tan\phi + x \sec^2\phi \, d\phi$

$$d\phi = \frac{-y}{s^2} dx + \frac{x}{s^2} dy$$

$$\nabla s = \frac{x}{s} \hat{x} + \frac{y}{s} \hat{y} = c_\phi \hat{x} + s_\phi \hat{y} = \hat{s}$$

$$\nabla \phi = \frac{-y}{s^2} \hat{x} + \frac{x}{s^2} \hat{y} = \frac{-s_\phi \hat{x} + c_\phi \hat{y}}{s} = \frac{\hat{\phi}}{s}$$

* formulas for vector derivatives in curvilinear coordinates

$$df = \frac{\partial f}{\partial q^i} dq^i = \frac{\partial f}{h_i \partial q^i} \cdot h_i dq^i = \nabla f \cdot d\vec{l}$$

$$\nabla f = \frac{df}{d\vec{r}} = \frac{\hat{e}_i}{h_i} \frac{\partial}{\partial q^i} f$$

$$d(\vec{A} \cdot d\vec{l}) = d(A_k h_k dq^k) = \frac{\partial}{\partial q^j} (h_k A_k) \, dq^j dq^k$$

$$= \varepsilon_{ijk} \frac{\partial (h_k A_k)}{h_j h_k \partial q^k} da_i = (\nabla \times \vec{A}) \cdot d\vec{a}$$

$$\nabla \times \vec{A} = \frac{d(\vec{A} \cdot d\vec{l})}{d^2\vec{r}} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 A^1 & h_2 A^2 & h_3 A^3 \end{vmatrix}$$

$$d(\vec{B} \cdot d\vec{a}) = d(B_i h_j dq^j h_k dq^k) = \frac{\partial}{\partial q^i} (h_j h_k B_i) \, dq^i dq^j dq^k$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q} \frac{\partial (h_j h_k B_i)}{\partial q^i} d\tau = \nabla \cdot \vec{B} \, d\tau$$

$$\nabla \cdot \vec{B} = \frac{d(\vec{B} \cdot d^2\vec{r})}{d^3\vec{r}} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i} (h_j h_k B_i)$$

i, j, k cyclic

this formula does not work for $\nabla^2 \vec{B}$

instead, use: $\nabla^2 = \nabla\nabla\cdot \; - \nabla\times\nabla\times$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q_i} f$$