Green Functions $G(x,x')$

* Green's functions are used to "invert" a differential operator
  ~ they solve a differential equation by turning it into an integral equation

* You already saw them last year! (in Phy 232)
  ~ the electric potential of a point charge

$$\S 1.5.1: \quad \nabla\cdot\frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

a) $\frac{1}{r^2} \to \infty$ at $r=0$ "singularity"

b) $\int_V \nabla\cdot\frac{\hat{r}}{r^2}\,d\tau = \oint_{\partial V} d\vec{a}\cdot\frac{\hat{r}}{r^2} = \oint_{\Omega} d\Omega\, r^2\frac{1}{r^2} = 4\pi$

independent of volume if 0 inside

thus $\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$

c) $\nabla\frac{1}{r} = \hat{r}\frac{\partial}{\partial r}\frac{1}{r} = -\frac{\hat{r}}{r^2}$

$$\boxed{\begin{array}{ccccc} V & \xrightarrow{-\nabla} & E & \xrightarrow{\nabla\cdot} & \rho/\varepsilon_0 \\ \frac{1}{r} & \rightarrow & \frac{\hat{r}}{r^2} & \rightarrow & 4\pi\delta^3(\vec{r}) \end{array}}$$

$$-\nabla^2 V = \rho/\varepsilon_0$$

(Poisson equation)

* Green's functions are the simplest solutions of the Poisson equation

$$G(\vec{r},\vec{r}') \equiv G(\mathscr{r}) = \frac{-1}{4\pi\mathscr{r}} = \nabla^{-2}\delta^3(\vec{\mathscr{r}})$$

  ~ is a special function which can be used to solve Poisson equation symbolically using the "identity" nature of $\delta^3(\vec{r}-\vec{r}') = \delta^3(\vec{\mathscr{r}})$

  ~ intuitively, it is just the "potential of a point source"

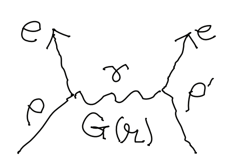
$$\nabla^2 G(\mathscr{r}) = \nabla\cdot\nabla\frac{-1}{4\pi\mathscr{r}} = \nabla\cdot\frac{\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = \delta^3(\vec{\mathscr{r}}) \qquad \vec{\mathscr{r}} \equiv \vec{r}-\vec{r}'$$

let $V = \int_V -G(\mathscr{r})\frac{\rho(\vec{r}')}{\varepsilon_0}\,d\tau'$ (solution to Poisson's eq.)

$$\nabla^2 V = \int_{V'} -\frac{\rho(\vec{r}')}{\varepsilon_0}\nabla^2 G(\vec{r}-\vec{r}')\,d\tau' = \int_{V'} -\frac{\rho(\vec{r}')}{\varepsilon_0}\delta^3(\vec{r}-\vec{r}')\,d\tau' = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

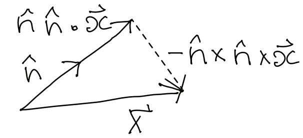
* this generalizes to one of the most powerful methods of solving problems in E&M
  ~ in QED, Green's functions represent a photon 'propagator'
  ~ the photon mediates the force between two charges
  ~ it 'carries' the potential from charge to the other

$$U = \int \rho V\,d\tau = \iint \rho G \rho'\,d\tau\,d\tau'$$



Section 1.6 - Helmholtz Theorem

* orthogonal projections $P_{\parallel}$ and $P_{\perp}$: a vector $\vec{n}$ divides the space X into $X_{\parallel n} \oplus X_{\perp n}$
geometric view: dot product $\hat{n}\cdot\vec{x}$ is length of $\vec{x}$ along $\hat{n}$
Projection operator: $P_{\parallel} \equiv \hat{n}\hat{n}\cdot$ acts on x: $P_{\parallel}\vec{x} = \vec{x}_{\parallel} = \hat{n}\hat{n}\cdot\vec{x}$



~ orthogonal projection: $\hat{n}\times$ projects $\perp$ to $\hat{n}$ and rotates by 90°

$\hat{x}_{\perp} = -\hat{n}\times(\hat{n}\times\hat{x}) = P_{\perp}\vec{x}$ $\quad P_{\perp} = -\hat{n}\times\hat{n}\times$ $\quad \boxed{P_{\parallel} + P_{\perp} = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$

* longtudinal/transverse separation of Laplacian (Hodge decomposition)

$\boxed{\begin{matrix}\nabla\cdot\vec{F} = \rho \\ \nabla\times\vec{F} = \vec{J}\end{matrix}}$ ~ is there a solution to these equations for $\vec{F}(r)$ given fixed source fields $\rho(\vec{r})$ and $\vec{J}(\vec{r})$? YES! (compare HW1 #1)

~ proof: $\nabla^2\vec{F} = \nabla\nabla\cdot\vec{F} - \nabla\times\nabla\times\vec{F}$ (longtudinal/transverse components of ∇)

~ formally, $\vec{F} = -\nabla\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho}}_{V}\Big) + \nabla\times\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}}_{\vec{A}}\Big)$ $\quad \rho, \vec{J}$ are SOURCES; $V, \vec{A}$ are POTENTIAL

~ what does ∇^{-2} mean? Note that $-\nabla^2\frac{1}{4\pi\mathscr{r}} = \delta^3(\vec{\mathscr{r}})$

~ thus $\nabla^{-2}\delta^3(\vec{\mathscr{r}}) = \frac{-1}{4\pi\mathscr{r}} \equiv G(\mathscr{r})$ (see next page) $\quad G = \frac{-1}{4\pi\mathscr{r}}$ is Green fn

~ use the δ-identity $\quad \rho(\vec{r}) = \int d\tau'\,\delta^3(\vec{\mathscr{r}})\,\rho(\vec{r}')$

$$V(\vec{r}) \equiv -\nabla^{-2}\rho(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{\mathscr{r}})\right)\rho(\vec{r}') = \int d\tau'\,\frac{\rho(\vec{r}')}{4\pi\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}}$$

$$\vec{A}(\vec{r}) \equiv -\nabla^{-2}\vec{J}(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{\mathscr{r}})\right)\vec{J}(\vec{r}') = \int d\tau'\,\frac{\vec{J}(\vec{r}')}{4\pi\mathscr{r}} = \frac{\mu_0}{4\pi}\oint\frac{I\,d\vec{l}}{\mathscr{r}}$$

~ thus any field can be decomposed into L/T parts $\quad \boxed{\vec{F} = -\nabla V + \nabla\times\vec{A}}$ with $V, \vec{A}$ defined above

SCALAR POTENTIAL V

* Theorem: the following are equivalent definitions of an "irrotational" field:

a) $\nabla\times\vec{F} = \vec{0}$ curl-less

b) $\vec{F} = -\nabla V$ where $V = \int\frac{d\tau'\,\nabla\cdot\vec{F}}{4\pi\mathscr{r}}$

c) $V(\vec{r}) = \int_{r_0}^{\vec{r}} -\vec{F}\cdot d\vec{l}$ is independent of path

d) $\oint\vec{F}\cdot d\vec{l} = 0$ for any closed path

* Gauge invariance:
if $\vec{F} = -\nabla V_1$ and also $\vec{F} = -\nabla V_2$
then $\nabla(V_2 - V_1) = 0$ and $V_2 - V_1 = V_0$ is constant
("ground potential")

VECTOR POTENTIAL $\vec{A}$

* Theorem: the following are equivalent definitions of a "solenoidal" field:

a) $\nabla\cdot\vec{F} = 0$ divergence-less

b) $\vec{F} = \nabla\times\vec{A}$ where $\vec{A} = \int\frac{d\tau\,\nabla\times\vec{F}}{4\pi\mathscr{r}}$

c) $? = \int_S\vec{F}\cdot d\vec{a}$ with ∂S fixed is independent of surface

d) $\oint\vec{F}\cdot d\vec{a} = 0$ for any closed surface

* Gauge invariance:
if $\vec{F} = \nabla\times\vec{A}_1$ and also $\vec{F} = \nabla\times\vec{A}_2$
then $\nabla\times(A_2 - A_1) = 0$ and $A_2 - A_1 = \nabla\lambda(\vec{r})$
("gauge transformation")