Section 2.3 - Electric Potential

* two personalities of a vector field: Flux $= \Phi_E = \int_S \vec{E}\cdot d\vec{a}$ (streamlines) through an area

Dr. Jekyl and Mr. Hyde Flow $= \mathcal{E}_E = \int_P \vec{E}\cdot d\vec{l}$ (equipotentials) downstream

* direct calculation of flow for a point charge

$$\mathcal{E}_E = \int_{\vec{r}=a}^{b} \vec{E}\cdot d\vec{l} = \int_{V'} \frac{dq'}{4\pi\varepsilon_0} \int_{\vec{r}=a}^{b} \frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2}$$

note: this is a perfect differential (gradient)

$$\frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2} = \frac{d\mathscr{r}}{\mathscr{r}^2} = d\frac{-1}{\mathscr{r}}$$

$$df = \nabla f \cdot d\vec{l}$$

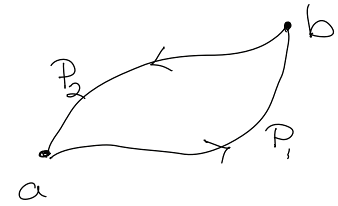
$$\nabla \mathscr{r} = \hat{\mathscr{r}}$$

$$= \int_{V'} \frac{dq'}{\varepsilon_0} \frac{1}{4\pi\mathscr{r}}\Big|_{\vec{r}=\vec{r}_a}^{\vec{r}_b} \equiv V(\vec{r})\Big|_a^b$$

~ open path: note that this integral is independent of path

thus $V(\vec{r}) \equiv -\mathcal{E}_E = \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$ is well-defined

by FTVC: $\Delta V = \int_{\vec{r}_0}^{\vec{r}} \nabla V \cdot d\vec{l}$ so $\boxed{\vec{E} = -\nabla V}$



~ ground potential $V(\vec{r}_0) = 0$ (constant of integration)

~ closed loop (Stokes theorem) $\mathcal{E}_E = \oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a} = 0 \iff \boxed{\nabla\times E = 0}$

for any surface S

* Poincaré lemma: if $\vec{E} = -\nabla V$ then $\nabla\times\vec{E} = -\nabla\times\nabla V = 0$

~ converse: if $\nabla\times\vec{E} = 0$ then $\vec{E} = -\nabla V$ so $\boxed{\vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0}$

* Poisson equation $\nabla\cdot\varepsilon_0 E = \boxed{-\nabla\cdot\varepsilon_0\nabla V = \rho}$ or $\nabla^2 V = \rho/\varepsilon_0$

~ next chapter devoted to solving this equation - often easiest for real-life problems

~ a scalar differential equation with boundary conditions on E_n or V

~ inverse (solution) involves: a) the solution for a point charge (Green's function)

$$V(\vec{r}) = \int_{V'} \frac{dq'}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}} = \int \frac{dq'}{\varepsilon_0} G(\vec{\mathscr{r}}) \quad \text{where} \quad G(\vec{\mathscr{r}}) = \frac{1}{4\pi\mathscr{r}}$$

$$\boxed{\begin{aligned}\nabla^2 G(\vec{\mathscr{r}}) &= \delta^3(\vec{\mathscr{r}})\\ G(\vec{\mathscr{r}}) &= \nabla^{-2}\delta^3(\vec{\mathscr{r}})\end{aligned}}$$

$$\nabla^2 G = \nabla\cdot\nabla\frac{1}{4\pi\mathscr{r}} = \nabla\cdot\frac{-\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = -\delta^3(\vec{\mathscr{r}})$$

b) an arbitrary charge distribution is a sum of point charges (delta functions)

$$\nabla^2 V = \int \frac{dq'}{\varepsilon_0}\nabla^2 G = \int_{V'} \frac{\rho(\vec{r}')d\tau'}{\varepsilon_0}\delta^3(\vec{\mathscr{r}}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\boxed{\rho(\vec{r}) = \int_{V'} \rho(\vec{r}')d\tau'\,\delta^3(\vec{r}-\vec{r}') = \int_{V'} dq'\,\delta^3(\vec{\mathscr{r}})}$$

going backwards:

$$V = \nabla^{-2}\frac{\rho(\vec{r})}{\varepsilon_0} = \int_{V'} \frac{\rho(\vec{r}')d\tau'}{\varepsilon_0}\nabla^{-2}\delta^3(\vec{\mathscr{r}}) = \int_{V'} \frac{dq'}{\varepsilon_0} G(\vec{\mathscr{r}})$$

~ this is an essential component of the Helmholtz theorem $\boxed{\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times}$

$$\vec{E} = \underbrace{-\nabla(-\nabla^{-2}\overbrace{\nabla\cdot E}^{\rho/\varepsilon_0})}_{V = -\nabla^{-2}\rho/\varepsilon_0 = \int_{V'}\frac{dq'}{4\pi\varepsilon_0\mathscr{r}}} + \nabla\times(-\nabla^{-2}\overbrace{\nabla\times\vec{E}}^{\vec{0}}) = -\nabla(-\nabla^{-2}\rho/\varepsilon_0) \quad \text{thus} \quad \vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0$$

* derivative chain $\boxed{V \xrightarrow{d} \vec{E} \xrightarrow{d} \rho}$

$$\begin{tikzcd}
V \arrow[r, shift left, "-\nabla V"] \arrow[rr, bend right, "\frac{d}{\int}", leftarrow, "\int\vec{E}\cdot d\vec{l}"'] & \vec{E} \arrow[l, shift left, "-\int\vec{E}\cdot d\vec{l}"] \arrow[r, shift left, "\nabla\cdot\varepsilon_0\vec{E}"] & \rho \arrow[l, shift left, "\int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}"]
\end{tikzcd}$$

$dq' = \rho(\vec{r}')d\tau'$ $\int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}$

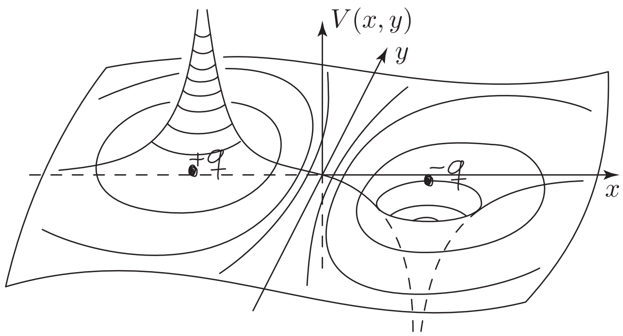
~ inverting Gauss' law is more tortuous path!

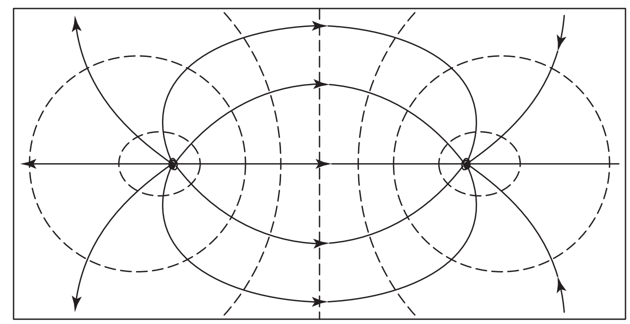
$$\rho \to V \to \vec{E} \qquad \vec{E} = -\nabla V = \int \frac{dq'}{4\pi\varepsilon_0}\nabla\frac{-1}{\mathscr{r}}$$

Field Lines and Equipotentials

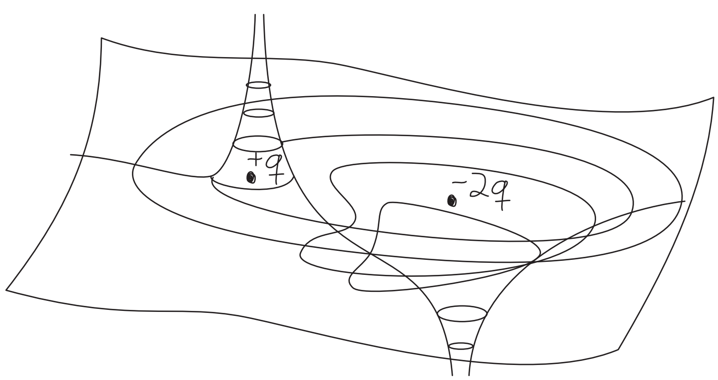
* for   along an equipotential surface:
  fo field lines are normal to equipotential surfaces

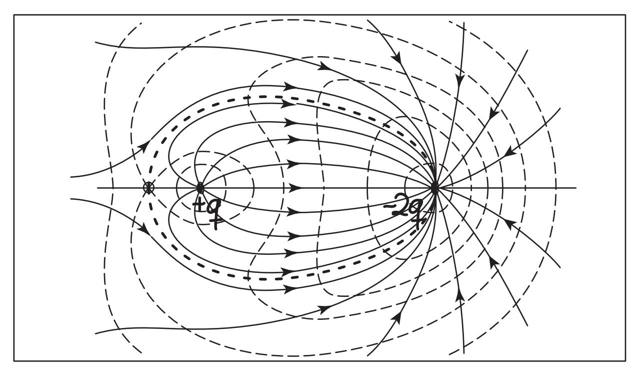
* dipole "two poles" – the word "pole" has two different meanings: (but both are relevant)
  a) opposite (+ vs – , N vs S, bi-polar)
  b) singularity (1/r has a pole at r=0)





* effective monopole (dominated by –2q far away)





* quadrupole (compare HW3 #2)