Section 2a - Examples

* show that $\nabla\cdot\vec{E} = \rho/\varepsilon_0$ from Coulomb's law

note that $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial(x-x')}, \frac{\partial}{\partial(y-y')}, \frac{\partial}{\partial(z-z')}\right) = \nabla_{\mathscr{r}}$ (if $\vec{x}'$ fixed)

$$\nabla\cdot\int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2} = \nabla\cdot\int_{V'}\frac{\rho(\vec{r}\,')\,d\tau'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_{V'}\rho(\vec{r}\,')\,d\tau'\,\nabla_{\mathscr{r}}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{r}\,')\,d\tau'\,4\pi\,\delta^3(\vec{\mathscr{r}}) = \rho(\vec{r})/\varepsilon_0$$

* derive Coulomb's law from the differential field equations

$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \quad \nabla\times\vec{E} = 0 \qquad \nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times$

$$\vec{E} = -\nabla\underbrace{\left(-\nabla^{-2}\nabla\cdot\vec{E}\right)}_{\vee} + \nabla\times\left(-\nabla^{-2}\,\cancelto{0}{\nabla\times\vec{E}}\right) = -\nabla\int\frac{d\tau'\,\nabla'\cdot\vec{E}(\vec{r}\,')}{4\pi\mathscr{r}} = -\nabla\int\frac{d\tau'\rho(\vec{r}\,')}{4\pi\varepsilon_0\mathscr{r}}$$

$$= \int\frac{d\tau'\rho(\vec{r}\,')}{4\pi\varepsilon_0}\nabla_{\mathscr{r}}\frac{-1}{\mathscr{r}} = \int\frac{d\tau'\rho(\vec{r}\,')}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}$$

* show that the differential and integral field equations are equivalent

$\Phi_E = Q/\varepsilon_0 \iff \nabla\cdot E = \rho/\varepsilon_0$

~ apply the divergence theorem

~ since Gauss' law holds for any volume, it is only true if the integrands are equal

$$\Phi_E = \oint_{\partial V} d\vec{a}\cdot\vec{E} = \int_V \nabla\cdot\vec{E}\,d\tau$$

$$Q/\varepsilon_0 = \int_V \rho/\varepsilon_0\,d\tau$$

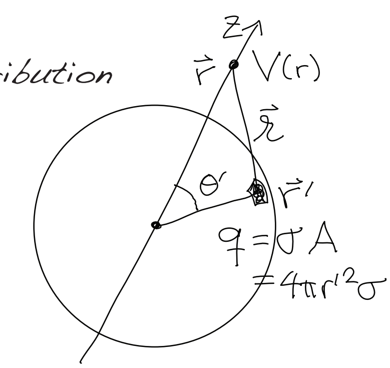
* Griffiths 2.6 find potential of spherical charge distribution

$$\int\vec{E}\cdot d\vec{a} = \int\rho/\varepsilon_0\,d\tau \qquad 4\pi r^2 E(r) = \begin{cases} q/\varepsilon_0 & \text{if } r>r' \\ 0 & \text{if } r<r' \end{cases}$$

$$\text{if } r>r' \quad V(r) = \int_\infty^r -\vec{E}\cdot d\vec{l} = \int_\infty^r \frac{-q\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\hat{r}\,dr = \frac{q}{4\pi\varepsilon_0}\left.\frac{+1}{r}\right|_\infty^r = \frac{q}{4\pi\varepsilon_0 r}$$

$$\text{if } r<r' \quad V(r) = V(r') + \int_{r'}^r -\vec{E}\cdot d\vec{l} = V(r') + \int_{r'}^r 0 = V(r')$$

* Griffiths 2.7 integrate potential due to spherical charge distribution



$$4\pi\varepsilon_0 V = \int_{sph.}\frac{\sigma\,da'}{\mathscr{r}}$$

$$= \int_{u=-1}^{1} 2\pi r'^2\sigma\,\frac{du}{\mathscr{r}}$$

$$= \frac{q}{2}\int_{u=-1}^{1}\frac{-d\mathscr{r}}{rr'}$$

$$= \frac{q}{2rr'}\left[-|r-r'| + |r+r'|\right]$$

$$= \frac{q}{2rr'}\begin{cases} -r+r'+r+r' & r>r' \\ +r-r'+r+r' & r<r' \end{cases}$$

$$da' = r'^2\,d\Omega'$$
$$= r'^2\sin\theta'\,d\theta'\,d\phi'$$
$$= r'^2\cdot -du\,d\phi'$$
$$u = \cos\theta'$$
$$-du = \sin\theta'\,d\theta'$$

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'u \Rightarrow (r\mp r')^2$$
$$2\mathscr{r}\,d\mathscr{r} = -2rr'\,du \qquad u = \pm 1$$

$$V(r) = \frac{q}{4\pi\varepsilon_0}\cdot\begin{cases} 1/r & \text{if } r>r' \\ 1/r' & \text{if } r<r' \end{cases}$$

* Griffiths 2.8 find the energy due to a spherical charge distribution

a) $W = \frac{1}{2}\int \sigma \cdot V = \frac{1}{2} q V = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 r'}$

b) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_{r=r'}^{\infty} r^2 dr\, d\Omega \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2$

$= \frac{q^2}{2\cdot 4\pi\varepsilon_0}\int_{r'}^{\infty}\frac{dr}{r^2} = \frac{q^2}{2\cdot 4\pi\varepsilon_0 r'}$

* Quiz: calculate field at origin from a hemispherical charge distribution

$$\vec{E} = \int \frac{dq\, \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\frac{q}{2\pi} d\Omega\, (-x\hat{x} - y\hat{y} - z\hat{z})}{4\pi\,\varepsilon_0\, R^3}$$

$$dq = \frac{q\, d\Omega}{2\pi} = \sigma\, da$$

$$= \frac{-q\hat{z}}{2\pi\cdot 4\pi\varepsilon_0 R^3} \underbrace{\int_{\theta=0}^{\pi/2} R\cos\theta\,(-d\cos\theta)}_{-R\frac{\cos^2\theta}{2}\Big|_0^{\pi/2} = -\frac{R}{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \frac{-q\hat{z}}{8\pi\varepsilon_0 R^2}$$