# Section 3.1 - Laplace's Equation

* overview: we leared the math (Ch 1) and the physics (Ch2) of electrostatics
  basically concepts of Phy 232 described in a new sophisticated language
  ~ Ch 3: Boundary Value Problems (BVP) with LaPlace's equation (NEW!)
  a) method of images   b) separation of variables   c) multipole expansion
  ~ Ch 4: Dielectric Materials: free and bound charge (more in-depth than Phy 232)

$$\chi \xrightarrow{d} (V,\vec{A}) \xrightarrow{d} (\vec{E},\vec{B}) \xrightarrow{d} 0$$
$$\varepsilon \| \mu \qquad \sigma$$
$$(\vec{D},\vec{H}) \xrightarrow{d} (\rho, J) \xrightarrow{d} 0$$

(I) Brute force!
$$\vec{E} = \int \frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2}$$

(II) Symmetry
$$\Phi_D = Q$$
$$\mathcal{E}_E = 0$$

(III) Elegant but cumbersome
$$\nabla\cdot\vec{D} = \rho \qquad \text{Ch.4}$$
$$\nabla\times\vec{E} = \vec{0}$$

(IV) Refined brute
$$V = \int \frac{dq'}{4\pi\varepsilon_0 \mathscr{r}}$$

(V) the WORKHORSE!!
$$-\nabla^2 V = \rho/_\varepsilon \qquad \text{Ch.3}$$

Equations of electrodyamics:

| | |
|---|---|
| $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ | Lorentz force |
| $\nabla\cdot\vec{J} + \partial_t\rho = 0$ | Continuity |
| $\nabla\cdot\vec{D} = \rho \quad \nabla\times\vec{E} + \partial_t\vec{B} = \vec{0}$ | Maxwell electric, |
| $\nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{H} - \partial_t\vec{D} = \vec{J}$ | magnetic fields |
| $\vec{D} = \varepsilon\vec{E} \quad \vec{B} = \mu\vec{H} \quad \vec{J} = \sigma\vec{E}$ | Constitution |
| $\vec{E} = -\nabla V - \partial_t\vec{A} \quad \vec{B} = \nabla\times\vec{A}$ | Potentials |
| $V \to V - \partial_t\lambda \quad \vec{A} \to \vec{A} + \nabla\lambda$ | Gauge transform |

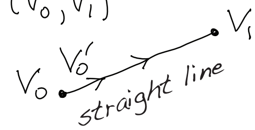
* Classical field equations - many equations, same solution:

| | | |
|---|---|---|
| Laplace/Poisson: | $\nabla^2 V = 0 \quad -\nabla\cdot\varepsilon\nabla V = \rho$ | ~ potentials $(V,\vec{A})$, dielectric $\varepsilon$, permeability $\mu$ |
| Maxwell wave: | $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(V,\vec{A}) - \nabla^2(V,\vec{A}) = \mu(\rho,\vec{J})$ | ~ speed of light $c = \frac{1}{\sqrt{\varepsilon\mu}}$, charge/current density $(\rho,\vec{J})$ |
| Heat equation: | $C\frac{\partial T}{\partial t} = k\nabla^2 T$ | ~ temp $T$, cond. $k$, heat $\vec{q} = -k\nabla u$, heat cap. $C$ |
| Diffusion eq: | $\frac{\partial u}{\partial t} = D\nabla^2 u$ | ~ concentration $u$, diffusion $D$, flow $D\nabla u$ |
| Drumhead wave: | $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f$ | ~ displacement $u$, speed of sound $c$, force $f$ |
| Schrödinger: | $\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial}{\partial t}\Psi$ | ~ prob amp $\Psi$, mass $m$, potential $V$, Planck $\hbar$ |

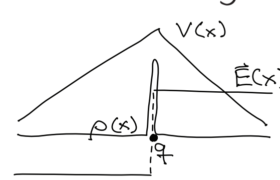
* 1-dimensional Laplace equation $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$

$$\frac{dV}{dx} = \int 0\,dx = a \qquad V = \int a\,dx = ax + b$$

~ $a, b$ satisfy boundary conditions $(V_0, V_0')$ or $(V_0, V_1)$
~ mean field: $V(x) = \frac{1}{2}\big(V(x-a) + V(x+a)\big)$
~ no local maxima or minima (stretches tight)



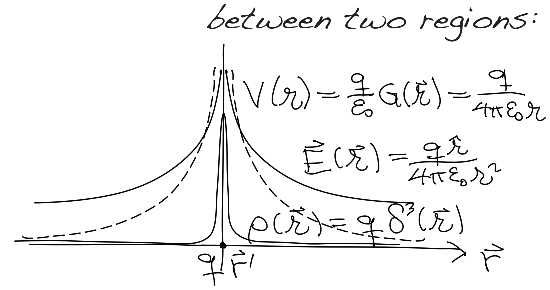

~ charge singularity
between two regions:




* 2-dimensional Laplace equation $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

~ no straighforward solution (method of solution depends on the boundary conditions)
~ Partial Differential Equation (elliptic 2nd order)
~ chicken & egg: can't solve $\frac{\partial^2 V}{\partial x^2}$ until you know $\frac{\partial^2 V}{\partial y^2}$
~ solution of a rubber sheet
~ no local extrema -- mean field: $V(\vec{r}) = \frac{1}{2\pi R}\oint_{\text{circle}} V\,dl$

~ charge singularity
between two regions:




* 3-dimensional Laplace equation $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

~ generalization of 2-d case
~ same mean field theorem: $V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$

# Boundary Conditions

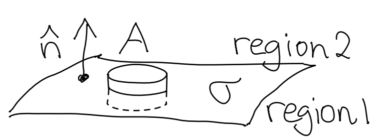
* 2nd order PDE's classified in analogy with conic sections: replacing $\frac{\partial}{\partial x}$ with $x$, etc
  a) Elliptic – "spacelike" boundary everywhere (one condition on each boundary point)
  eg. Laplace's eq, Poisson's eq. $\nabla^2 V = 0$ $\quad -\nabla\cdot\varepsilon\nabla V = \rho$
  b) Hyperbolic – "timelike" (2 initial conditions) and "spacelike" parts of the boundary
  eg. Wave equation $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(V,\vec{A}) - \nabla^2(V,\vec{A}) = \mu(\rho,\vec{J})$
  c) Parabolic – $1^{st}$ order in time (1 initial condition)
  eg. Heat equation, Diffusion equation $C\frac{\partial T}{\partial t} = k\nabla^2 T$ $\quad \frac{\partial u}{\partial t} = D\nabla^2 u$

* Uniqueness of a BVP (boundary value problem) with Poisson's equation:
  if $V_1$ and $V_2$ are both solutions of $\nabla^2 V = -\rho/\varepsilon_0$ then let $U = V_1 - V_2$ $\quad \nabla^2 U = 0$
  integration by parts: $\nabla\cdot(U\nabla U) = U\,\nabla\cdot\nabla U + \nabla U\cdot\nabla U = U\nabla^2 U + (\nabla U)^2$
  in region of interest: $\oint_{\partial V} d\vec{a}\cdot(U\nabla U) = \int_V \nabla\cdot(U\nabla U)\,d\tau = \int_V U\nabla^2 U + (\nabla U)^2\,d\tau$
  note that: $\nabla^2 U = 0$ and $(\nabla U)^2 > 0$ always
  thus if $\int_{\partial V} d\vec{a}\cdot U\nabla U = \int_{\partial V} da\, \underbrace{U}_{(a)}\, \underbrace{\frac{\partial U}{\partial n}}_{(b)} = 0$ then $\int(\nabla U)^2 d\tau = 0 \Rightarrow U = 0$ everywhere
  a) Dirichlet boundary condition: $U = 0$ – specify potential $V_1 = V_2$ on boundary
  b) Neuman bounary condition: $\frac{\partial U}{\partial n} = 0$ – specify flux $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ on boundary

* Continuity boundary conditions – on the interface between two materials

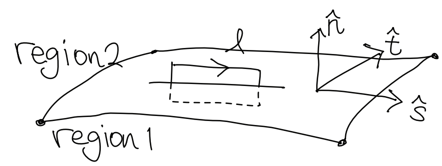
Flux:

$\vec{D} \equiv \varepsilon\vec{E}$ (shorthand for now)




$$\Phi = \oint_{\partial V} \vec{D}\cdot d\vec{a} = \int_V \sigma\, da = Q$$
$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1)A = \sigma\cdot A$$
$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma$$
$$-\frac{\partial V_2}{\partial n} + \frac{\partial V_1}{\partial n} = \sigma/\varepsilon_0$$

Flow:




$\hat{s}\times\hat{t} = \hat{n}$

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a}$$
$$\hat{s}\cdot(\vec{E}_2 - \vec{E}_1)\,l = \hat{t}\cdot\nabla\times\vec{E}\; l w = 0$$
$$\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0$$
$$V_2 = V_1$$

* the same results obtained by integrating field equations across the normal

$$\nabla\cdot\vec{D} = \rho/\varepsilon_0 \qquad\qquad \nabla\times\vec{E} = \vec{K}_e\,\delta(n) \qquad \nabla\times\vec{E} = \begin{vmatrix} \hat{s} & \hat{t} & \hat{n} \\ \partial_s & \partial_t & \partial_n \\ E_s & E_t & E_n \end{vmatrix}$$

$$\int_-^+ dn\left(\frac{\partial D_n}{\partial n} + \frac{\partial D_s}{\partial s} + \frac{\partial D_t}{\partial t}\right) = \int_-^+ dn\,\sigma\,\delta(n) \qquad \int_-^+ dn\left(\hat{t}\,\frac{\partial E_s}{\partial n} - \hat{s}\,\frac{\partial E_t}{\partial n}\right) = \int_-^+ dn\,\vec{K}_e\,\delta(n)$$

$$\int dD_n = \hat{n}\cdot\Delta\vec{D} = \sigma \qquad\qquad \hat{n}\times\Delta\vec{E} = \vec{K}_e = 0$$

~ opposite boundary conditions for magnetic fields: $\hat{n}\cdot\Delta\vec{B} = 0$ $\quad \hat{n}\times\Delta\vec{H} = \vec{K}$