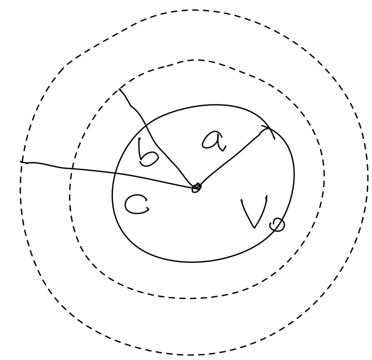
Section 3.2 - Method of Images

* concept: in a region $\mathcal{R}$, $V(\vec{r})$ depends ONLY on the boundary of V at $\partial\mathcal{R}$
  ~ it doesn't matter how it was created, or where charge is outside $\mathcal{R}$
  ~ more than one charge distribution can generate the same $V(\vec{r})$ inside $\mathcal{R}$

* Example 1: $V=V_0$ in a constant sphere of radius a



$V=\frac{q}{4\pi\varepsilon_0 r}$ for a point charge or any spherically symmetric charge distrubtion of total charge q inside r

i) $q_0 = 4\pi\varepsilon_0 a \cdot V_0$ is one solution

ii) $q_1 = 4\pi\varepsilon_0 b\, V_0$ is another solution

iii) how about $+q$ at radius b and $-q$ at radius c?

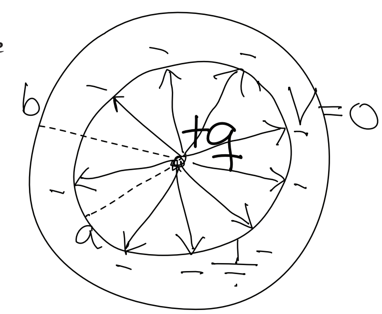
$r>c$, $V = \frac{+q}{4\pi\varepsilon_0 r} + \frac{-q}{4\pi\varepsilon_0 r} = 0$ no charge at ∞

and for $b>r>c$ $V=\frac{q}{4\pi\varepsilon_0 b} - \frac{q}{4\pi\varepsilon_0 c} = V_0 = \frac{q_0}{4\pi\varepsilon_0 a}$

so $q = \frac{q_0}{a}\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}$ for example, if $\begin{matrix} b=a \\ c=2a \end{matrix}$ then $q=2q_0$

in the case, the nonzero E between b and c, and builds up the potential at a

* Example 2a: point charge $+q$ at the center of a grounded sphere
  ~ assume a shell of charge q' at radius b, where $b>a$



$$V(a) = \frac{q}{4\pi\varepsilon_0 a} + \frac{q'}{4\pi\varepsilon_0 b}$$ so $q' = -\frac{b}{a} q$

for example $-q$ at $r=a$

* Example 3a: dipole: point charge $+q$ at $z=d$ and $-q$ at $z=-d$

$$V(z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

~ note that $V(z=0)=0$ so we can form a boundary value problem for $z>0$, $V(z=0)=0$ with the same solution!

~ induced surface charge: $\sigma = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial n} = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$ let $\begin{matrix} u=s^2+d^2 \\ du=2s\,ds \end{matrix}$

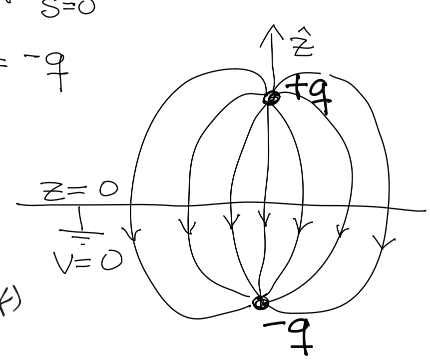
total induced charge: $$Q = \int \sigma\, da = \int_{\phi=0}^{2\pi}\int_{s=0}^{\infty} \sigma\, s\, ds\, d\phi = \frac{-2\pi q d}{2\pi}\int_{s=0}^{\infty}(s^2+d^2)^{-3/2}$$

$$= qd \cdot -u^{-1/2}\Big|_{s=0}^{\infty} = -q$$

~ force on charge:

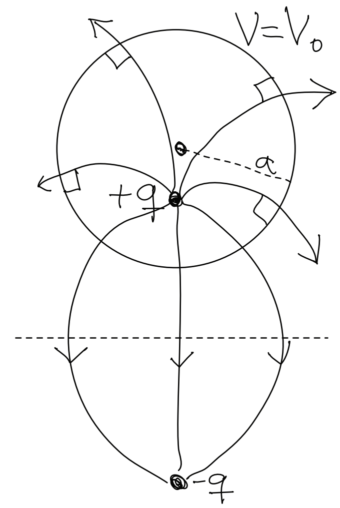
$$\vec{F} = q\vec{E} = -q\nabla V = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

~ energy in the system: $$W = \frac{1}{2}\left(W_0\right) = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$$



this is only half the value of dipole problem, because the induced charge is brought into zero potential (no work)

* Example 2b=3b: Point charge inside a conducting sphere
  - ~ move the point charge away from the center in Example 2a
  - ~ choose a different equipotential surface in Example 3a
  - ~ you solved a similar problem in cylindrical coorindates (HW5 #4b, continued in HW6 #4)



* Example 4: Coefficients of potential / capacitance (see notes, Section 2.5)

  see also Griffiths, Section 3.1.6 for uniqueness theorem applied to conductors.
  - ~ let a system of conductors C_1, C_2, ... C_n be contained in a region R held at $V=0$.
  - ~ there is a unique solution to the boundary value problem (BVP) with total charge Q_1, Q_2, ... Q_n on each of the conductors.
  - ~ the potential on each condutor is a linear combination of the charge on every conductor

    proof: superposition of solutions with unit charge on each conductor