Section 3.3.1 - Separation of Variables (Cartesian)

* goal: solve Lapalce's equation (a single PDE) by converting it into one ODE per variable
method: separate the equation into separate terms in x, y, z
start by factoring the solution $V(x,y,z) = X(x)\,Y(y)\,Z(z)$
trick: if $f(x) = g(y)$ where $f(x)$ is independent of y and $g(y)$ is independent of x
then they must both constant
endgame: form the most possible general solution as a linear combination of
all possible products of solutions in each variable.
Solve for unique values of the coefficients using the boundary conditions
analogy: the set of all solutions forms a vector space
the basis vectors are independent individual solutions

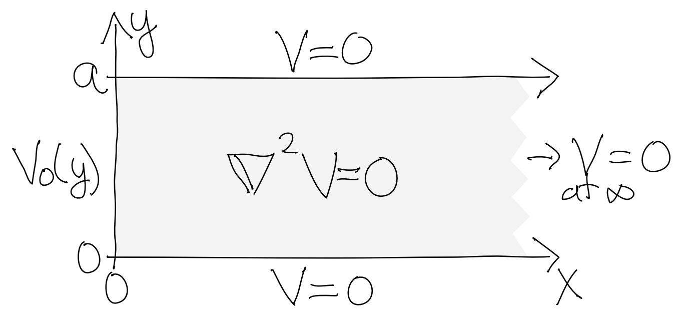
* Example: semi-infinite strip with non-zero voltage at one end

$$V(x,y) = X(x)\cdot Y(y)$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)=k^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)=-k^2} = 0$$

$$X'' - k^2 X = 0 \qquad Y'' + k^2 Y = 0$$

$$X = Ae^{kx} + Be^{-kx} \qquad Y = C\sin(ky) + D\cos(ky)$$



~ boundary conditions (BC):

1) $V(\infty,y)=0 \Rightarrow A=0$

2) $V(x,0)=0 \Rightarrow D=0$

3) $V(x,a)=0 \Rightarrow \sin(ka)=0 \quad k_n=\frac{n\pi}{a}$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-k_n x}\sin(k_n y)$$

4) $V_0(y) = \sum_{n=1}^{\infty} C_n \sin(k_n y)$

$$\int_0^a \sin(k_m y)\,V_0(y)\,dy = \sum_{n=1}^{\infty} C_n \int_0^a \sin(k_n y)\cdot\sin(k_m y)\,dy$$

$$= \sum_{n=1}^{\infty} C_n \int_0^a \cos\left(\tfrac{(n-m)\pi}{a}y\right) - \cos\left(\tfrac{(n+m)\pi}{a}y\right)dy.$$

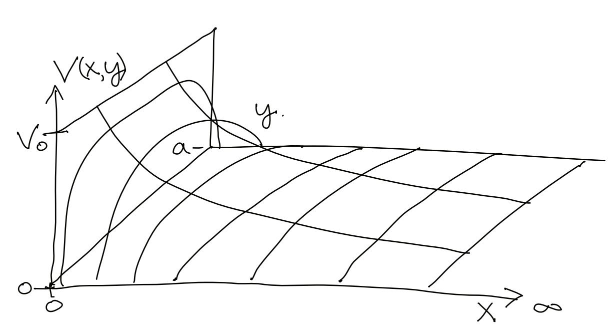
$$= \sum_{n=1}^{\infty} C_n \frac{a}{2}\delta_{nm} \qquad = \frac{a}{2} C_m$$

~ if $V_0(y) = \text{const} = V_0$ then

$$C_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) V_0\,dy = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ odd}\end{cases}$$

$$V(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

(Fourier decomposition)



* Vector Analogy:

$$\hat{e}_i \cdot \vec{V} = \hat{e}_i \cdot (V_j \hat{e}_j)$$
$$= V_j \delta_{ij} = V_i$$

$$\hat{x}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = a$$
$$\hat{y}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = b$$
$$\hat{z}\cdot(a\hat{x}+b\hat{y}+c\hat{z}) = c$$

$$\phi_n(x) = \sin(k_n x) \qquad V(x) = \sum_{n=1}^{\infty} C_n \phi_n(x)$$
$$\langle \phi_n | \phi_m \rangle = \int_0^a \sin(k_n x)\cdot\sin(k_m x)\,dx = \frac{a}{2}\delta_{nm}$$
$$C_m = \langle \phi_m(x) | V(x) \rangle \big/ \frac{a}{2}$$