Section 3.4 - Multipoles

* binomial expansion

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

* Pascal's triangle

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1

$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$

~ general form

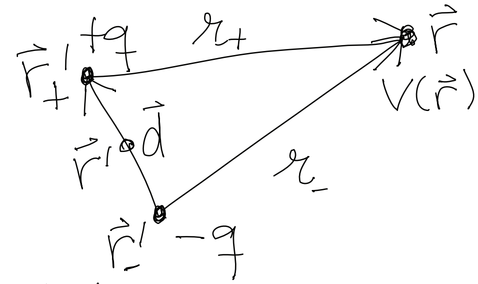
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}$$

~ if $n \to \alpha$ (any real number), then the series does not terminate

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

~ example: $$\frac{1}{1-x} = 1-(-x) + \frac{-1\cdot -2}{1\cdot 2}(-x)^2 + \frac{-1\cdot -2\cdot -3}{1\cdot 2\cdot 3}(-x)^3 + \ldots$$
$$= 1 + x + x^2 + x^3 + \ldots$$ for radius of convergence $|x|<1$

* 2-pole expansion



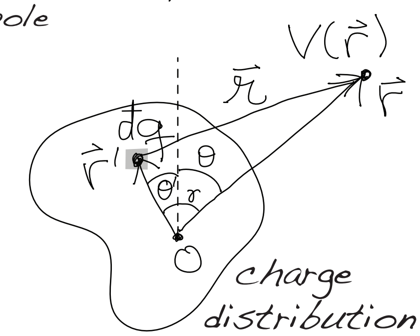
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$\vec{r}_\pm = \vec{r} \mp \tfrac{1}{2}\vec{d} \qquad r_\pm^2 = r^2 \mp r d\cos\theta + \tfrac{1}{4}d^2$$

$$\frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta + \ldots\right)$$

$$V(\vec{r}) = \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \boxed{\vec{p} = q\vec{d}}$$ electric dipole moment

* general axial-symmetric multipole expansion



$$r^2 = (\vec{r}-\vec{r}')^2 = r^2\left(1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\right) \equiv r^2(1+\varepsilon)$$

$$\frac{1}{r} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \tfrac{1}{2}\varepsilon + \tfrac{3}{8}\varepsilon^2 - \tfrac{5}{16}\varepsilon^3 + \ldots\right)$$

$$= \frac{1}{r}\left(1 + \frac{r'}{r}\cos\gamma + \frac{r'^2}{r^2}\frac{1}{2}(3\cos^2\gamma - 1) + \frac{r'^3}{r^3}\frac{1}{2}(5\cos^3\gamma - 3\cos\gamma)\right) + \ldots$$

$$= \frac{1}{r}\left(P_0(\cos\gamma) + \frac{r'}{r}P_1(\cos\gamma) + \frac{r'^2}{r^2}P_2(\cos\gamma)\right) + \cdots = \boxed{\sum_{\ell=0}^{\infty}\frac{r'^\ell}{r^{\ell+1}}P_\ell(\cos\theta)\,P_\ell(\cos\theta')}$$

(addition formula)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell+1}}P_\ell(\cos\theta)\underbrace{\int dq'\, r'^\ell P_\ell(\cos\theta')}_{Q^{(\ell)}_{int}}$$

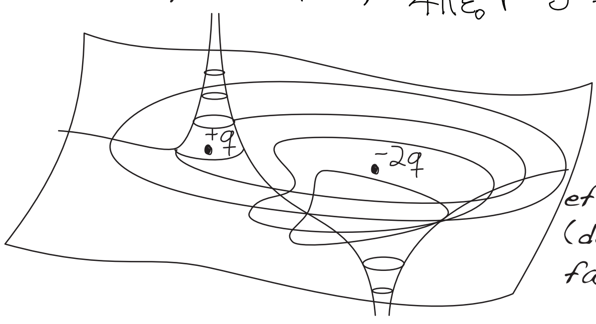
multipole potential

$Q^{(\ell)}_{int}$ electric multipole (monopole, dipole, quadrupole)

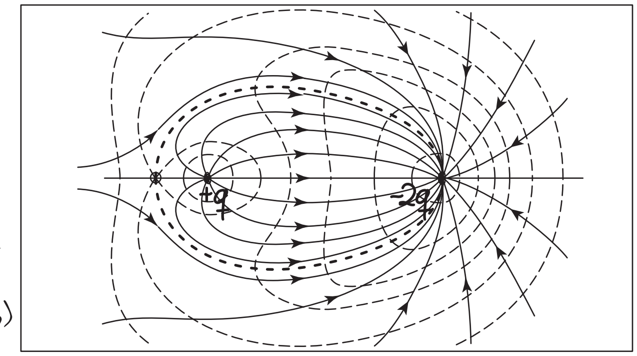
~ $Q^{(\ell)}_{int}$ are coefficients of the general solution of Laplace equation in spherical coords

* *monopole* $\quad V(\vec r) \approx \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r}\int dq' = \dfrac{q}{4\pi\varepsilon_0 r}$

$q \equiv \int dq'$

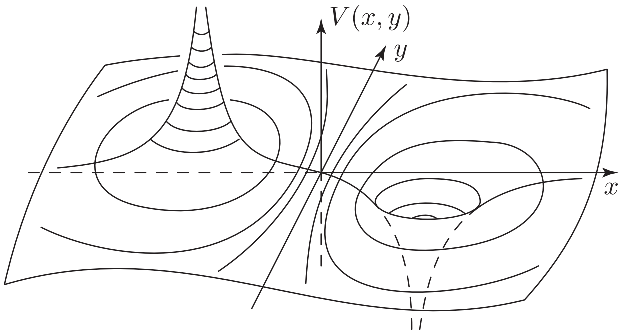


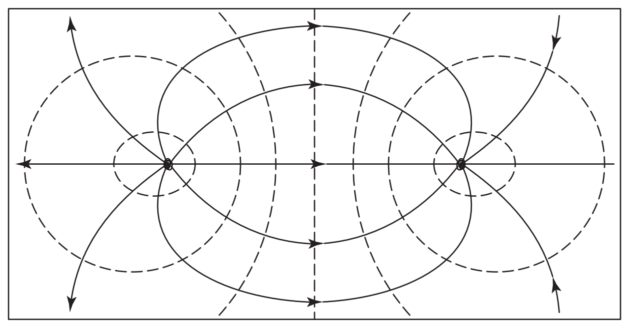
effective monopole (dominated by -2q far from the origin)



* *dipole* $\quad V_1(\vec r) = \dfrac{1}{4\pi\varepsilon_0 r^2}\int dq'\, r'\cos\theta = \dfrac{\vec p\cdot\vec r}{4\pi\varepsilon_0 r^3} \qquad \vec p = \int dq'\, \vec r\,'$

if $q = \int dq' = 0$ *then* $T_{\vec a}[\vec p] = \int dq'(\vec r\,' - \vec a) = \int dq'\,\vec r\,' - \vec a \overbrace{\int dq'}^{0} = \vec p$





* *quadrupole*

$$V_2(\vec r) = \frac{1}{4\pi\varepsilon_0 r^3}\int dq'\, r'^2 \tfrac{1}{2}(3\cos^2\theta' - 1) = \frac{1}{4\pi\varepsilon_0 r^5}\int dq' \tfrac{1}{2}\left(3(\vec r\cdot\vec r\,')^2 - r^2\right) \qquad Q_{xx}+Q_{yy}+Q_{zz}=0$$

$$= \frac{\tfrac{1}{2}\vec r\cdot\overleftrightarrow{Q}\cdot\vec r}{4\pi\varepsilon_0 r^5}$$

$$\overleftrightarrow{Q} = \int dq'(3\vec r\,'\vec r\,' - I r'^2) = \int dq' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2 - r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2 - r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq'(3r_i' r_j' - \delta_{ij} r'^2)$$

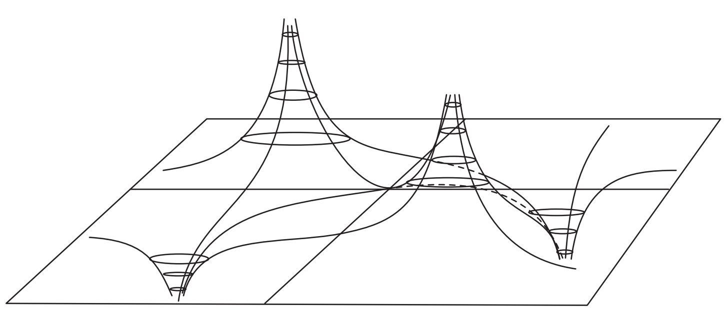
$$T_{\vec a}[\overleftrightarrow{Q}] = \int dq'\, 3(\vec r\,' - \vec a)(\vec r\,' - \vec a) - (\vec r\,' - \vec a)^2 I$$

$$= \int dq'(3\vec r\,'\vec r\,' - r'^2 I) - 3(\vec r\,'\vec a + \vec a\vec r\,' - \vec a\vec a) + (2\vec r\,'\cdot\vec a + \vec a^2)I$$

$$= \overleftrightarrow{Q}_0 - [3(\vec p\vec a + \vec a\vec p) - 2\vec p\cdot\vec a I] + [3\vec a\vec a - \vec a^2 I]q$$

$$\overleftrightarrow{Q} = 3\sum_i \vec p_i\vec a_i + \vec a_i\vec p_i - 2\vec p_i\cdot\vec a_i I$$

dipoles $\vec p_i$ at positions $\vec a_i$



sepratrix (potentials) $\quad$ *sepratrix (field lines)*