Section 3.4 - Multipoles (continued)

* spherical solutions $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$ solving Laplace equation $\nabla^2 V = 0$

* internal multipole $Q^{(l)}_{int} = B_l$

$r \to \infty$: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} Q^{(l)}_{int} \frac{1}{r^{l+1}} P_l(\cos\theta)$

$r' \to 0$: $Q^{(l)}_{int} = \int dq'\, r'^l P_l(\cos\theta')$

	$l=0$	1	2	3
V	$1/r$	$1/r^2$	$1/r^3$	$1/r^4$
E	$1/r^2$	$1/r^3$	$1/r^4$	$1/r^5$

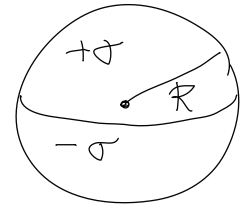
* external multipole $Q^{(l)}_{ext} = A_l$

$r \to 0$: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} Q^{(l)}_{ext}\, r^l P_l(\cos\theta)$

$r' \to \infty$: $Q^{(l)}_{ext} = \int dq' \frac{1}{r'^{l+1}} P_l(\cos\theta')$

	$l=0$	1	2	3
V	const	r	r^2	r^3
E	—	const	r	r^2

* example: calculate the dipole moment of two oppositely charge hemispheres



$\sigma = \frac{q}{2\pi R^2}$

$\vec{p} = \int dq'\, \vec{r}'$ $\quad p_x = p_y = 0$

$p_z = \int_{\theta=0}^{\pi} \sigma' da' \cdot z' = \int_{x=-1}^{1} \sigma \cdot 2\pi R^2 dx\, Rx$

$= \int_{x=-1}^{0} \frac{-q}{2\pi R^2} 2\pi R^2 dx\, Rx + \int_{x=0}^{1} \frac{q}{2\pi R^2} 2\pi R^2 dx\, Rx = qR\left[\int_{-1}^{0} x dx + \int_0^1 x dx\right] = qR$

* example: internal and external moments of a 4-pole $\quad +q$ @ (a,a) $\quad -q$ @ $(a,-a)$ $\quad -q$ @ $(-a,a)$ $\quad +q$ @ $(-a,-a)$

$q = \sum_i q_i = q + q - q - q = 0$

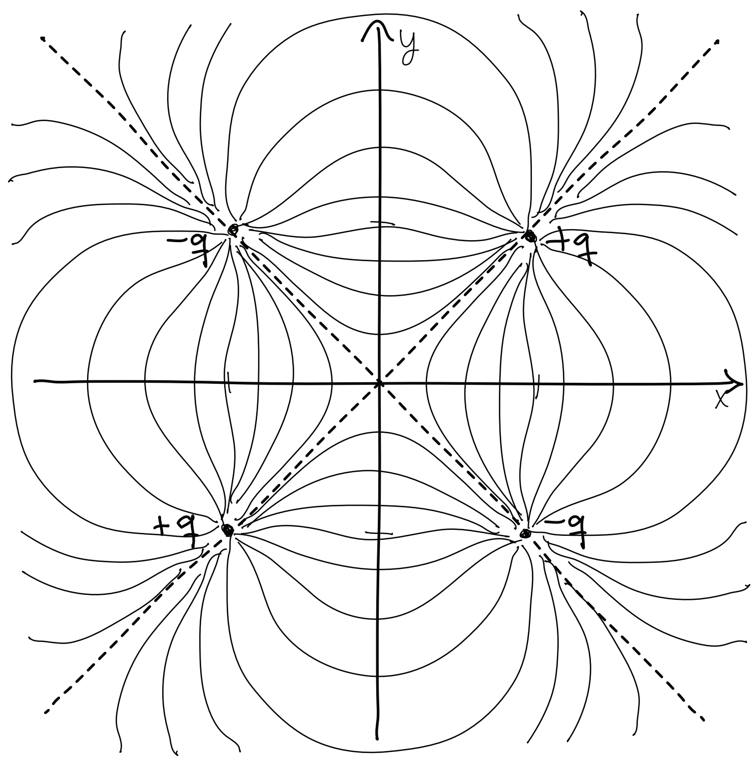
$\vec{p} = \sum_i q_i \vec{r}_i = q(a,a) - q(-a,a) - q(a,-a) + q(-a,-a) = \vec{0}$

$Q_{zx} = \sum_i 3 q_i z_i x_i = 0 = Q_{zy}$

$Q_{xy} = \sum_i 3 q_i x_i y_i$
$= +3q \cdot a \cdot a - 3q(-a)(a) - 3q(a)(-a) + 3q(-a)(-a) = 4qa^2$

$Q_{xx} = \sum_i q_i (3x_i'^2 - r'^2)$
$= q(3a^2 - 2a^2) - q(3a^2-2a^2) - q(3a^2-2a^2) + q(3a^2-2a^2) = 0$

$Q_{zz} = \sum_i q_i (3z_i^2 - r_i^2)$
$= (q - q - q + q)(0 - 2a^2) = 0$



* *electric field of a dipole* $V = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} + \frac{-1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos\theta\,\hat{r} - \hat{z}\right) = \frac{3\,\vec{p}\cdot\hat{r}\,\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$$