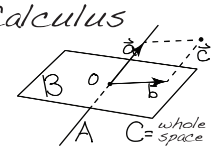
Review of Electrostatics (Chapters 1-4)

* Chapter 1: Mathematics - Vector Calculus

~ Vectors (it's all about being Linear!)
  linear combinations; projections
  basis (independence, closure)



~ Metric & Cross Product (Bilinear)
  orthonormal basis
  longitudinal / transverse projections

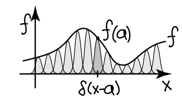
$$\vec{x} = \vec{b}_i x^i \equiv \sum_{i=1}^{3} \vec{b}_i x^i$$

~ Linear Operators
  eigenstuff: rotations / stretches

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \qquad \sum_{i=1}^{3} \hat{e}_i \hat{e}_i \cdot = \mathbb{I}$$

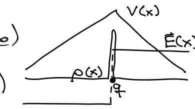
~ Function Spaces - continuous vs discrete
  Sturm-Liouville (orthogonal eigenfunctions)
~ Vector Derivatives and Integrals (Linearization)
  Natural ordering by dimension

~ Delta function (pole)



$$a_i = \sum_{j=1}^{n} \delta_{ij} a_j \qquad f(x) = \int_{-\infty}^{\infty} dx'\, \delta(x-x')\, f(x')$$

~ Greens function (tent/pole)



$$\nabla^2 G(\mathbf{r}) = \nabla^2 \frac{-1}{4\pi r} = \nabla \cdot \frac{\hat{r}}{4\pi r^2} = \delta^3(\mathbf{r})$$

~ Helmholtz theorem: source and potential

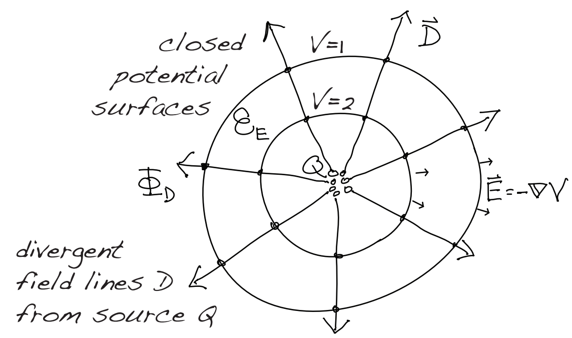
$$\vec{F} = -\nabla\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho}}_{V}\Big) + \nabla\times\Big(\underbrace{-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}}_{\vec{A}}\Big)$$

$$= -\nabla V + \nabla\times\vec{A} \qquad \nabla^2(V, \vec{A}) = -(\rho, \vec{J}) \qquad \begin{matrix}\nabla\cdot\vec{F} = \rho \\ \nabla\times\vec{F} = \vec{J}\end{matrix}$$

RANK	REGION	INTEGRAL	THEOREM	DERIVATIVE	GEOMETRY
scalar	∂ Q point	$\Delta f = f\|_a^b$ change	$f\|_a^b = \int_a^b \nabla f \cdot d\vec{l}$ ×	$df = \nabla f \cdot d\vec{l}$ ×	level surface d
vector	∂ P path	$\mathcal{E}_A = \oint_{\partial S} \vec{A}\cdot d\vec{l}$ flow	$\oint_{\partial S} \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{a}$	$d\vec{A}\cdot d\vec{l} = \nabla\times\vec{A}\cdot d\vec{a}$	flow sheets d
p-vector	∂ S surface	$\Phi_B = \oint_{\partial V} \vec{B}\cdot d\vec{a}$ flux	$\oint_{\partial V} \vec{B}\cdot d\vec{a} = \int_V \nabla\cdot\vec{B}\, d\tau$	$d\vec{B}\cdot d\vec{a} = \nabla\cdot\vec{B}\, d\tau$	flux tubes d
p-scalar	∂ V volume	$Q_\rho = \int_V \rho\, d\tau$ subst.	×	×	subst boxes d

* Chapter 2: Formulations of Electrostatics

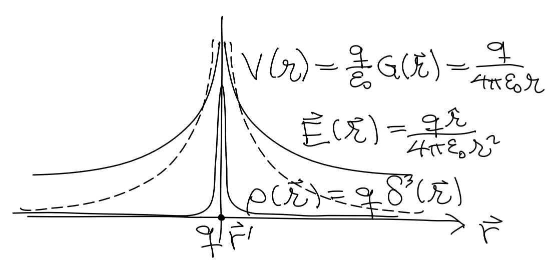
Integral	Differential	Boundary
$\vec{E} = \int \frac{dq\, \hat{r}}{4\pi\varepsilon_0 r^2}$	$\nabla^2 \vec{E} = -\nabla\rho/\varepsilon_0$	$dq = \sigma\, da$
$\Phi_D = Q$ $\mathcal{E}_E = 0$	$\nabla\cdot\vec{D} = \rho$ $\nabla\times\vec{E} = 0$	$\Delta\hat{n}\cdot\vec{D} = D_{2n} - D_{1n} = \sigma_f$ $\Delta\hat{n}\times\vec{E} = E_{2t} - E_{1t} = 0$
$V = \int \frac{dq'}{4\pi\varepsilon_0 r}$ $V = -\int \vec{E}\cdot d\vec{l}$	$\nabla^2 V = -\rho/\varepsilon_0$ $\vec{E} = -\nabla V$	$-\varepsilon_2 \partial_n V_2 + \varepsilon_1 \partial_n V_1 = \sigma_f$ $V_2 - V_1 = 0$



Relation between potential, field, and source:

$$V \underset{-\int \vec{E}\cdot d\vec{l}}{\overset{-\nabla V}{\rightleftarrows}} \vec{E} \underset{\int \frac{dq'\,\hat{r}}{4\pi\varepsilon_0 r^2}}{\overset{\nabla\cdot\varepsilon_0\vec{E}}{\rightleftarrows}} \rho \qquad dq' = \rho(\vec{r}')\,d\tau'$$

$$\int \vec{E}\cdot d\vec{l} \qquad \frac{d}{\int} \qquad \int \frac{dq'\,\hat{r}}{4\pi\varepsilon_0 r^2}$$



~ Work and Electric field energy: flux × flow

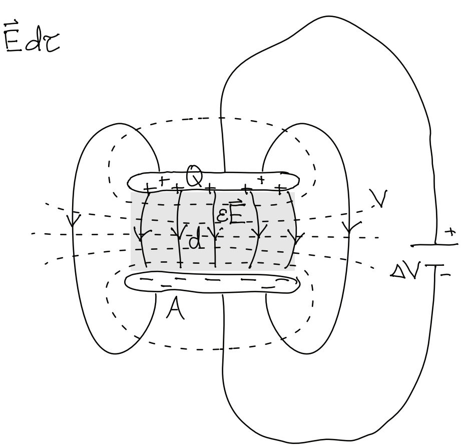
$$\vec{F} = q\vec{E} \qquad \vec{F} = m\vec{g}$$
$$W = q\underbrace{Ed}_{\text{potential}=V} \qquad W = m\underbrace{gh}_{\text{potential danger}}$$

$$\Delta W = \int \Delta\rho_f V\, d\tau = \oint d\vec{a}\cdot(\Delta\vec{D}\, V) - \int \Delta\vec{D}\cdot\vec{E}\, d\tau$$

$$W = \int \tfrac{1}{2}\vec{D}\cdot\vec{E}\, d\tau \quad \text{field energy density}$$

~ conductors and capacitance: flux / flow

$\rho = E = 0$, V=const inside; $D = \sigma$, V laminar outside conductor



$$C = \frac{Q}{\Delta V} = \frac{\Phi_D}{\mathcal{E}_F} \qquad Q = \int d\vec{a}\cdot\vec{D} = \Phi_D \quad \text{(closed surface)}$$

$$= \frac{\varepsilon\Phi_E}{\mathcal{E}_r} \approx \frac{\varepsilon A}{d} \qquad \Delta V = \int d\vec{l}\cdot\vec{E} = \mathcal{E}_E \quad \text{(open path)}$$

=

* Chapter 3: Solutions of LaPlace Equation

~ Uniqueness Theorem for exterior boundary conditions

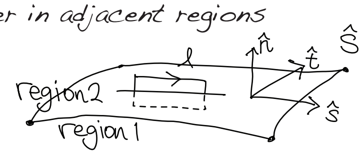
$$0 = \oint_{\partial V} da\, \underbrace{U}_{(a)}\, \underbrace{\frac{\partial U}{\partial n}}_{(b)} = \oint_{\partial V} d\vec{a}\cdot(U\nabla U) = \int_V \nabla\cdot(U\nabla U)\,d\tau = \int_V U\nabla^2 U + (\nabla U)^2\,d\tau$$

a) Dirichlet B.C. specifies potential on boundary; b) Neuman B.C. specifies flux on boundary

~ continuity boundary conditions stitch potentials together in adjacent regions

Flux: $\vec{D} \equiv \varepsilon\vec{E}$

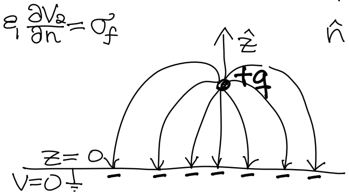
Flow:



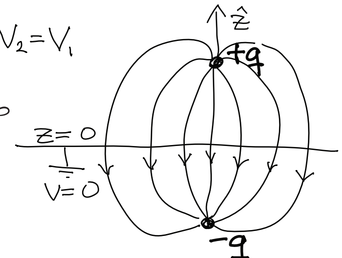
$\hat{s}\times\hat{t} = \hat{n}$

$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma \qquad -\varepsilon_2\frac{\partial V_2}{\partial n} + \varepsilon_1\frac{\partial V_2}{\partial n} = \sigma_f \qquad \hat{n}\times(\vec{E}_2-\vec{E}_1) = 0 \qquad V_2 = V_1$

A) METHOD OF IMAGES
find a point charge distribution
with the same B.C.'s
same solution by uniques theorem



is equivalent to



B) METHOD OF SEPARATION OF VARIABLES
separate Laplacian (10 known coordinate systems)
solve Sturm-Louiville ODE in each dimension
match boundary conditions to find coefficients
Fourier trick : orthogonal basis functions

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{\vec{k}\cdot\vec{r}} = \sum_{n=1}^{\infty} C_n e^{-k_n x}\sin(k_n y)$$

$$\phi_n(x) = \sin(k_n x) \qquad V(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

$$\langle\phi_n|\phi_m\rangle = \int_0^a \sin(k_n x)\cdot\sin(k_m x)\,dx = \frac{a}{2}\delta_{nm}$$

$$C_m = \langle\phi_m(x)|V(x)\rangle / \frac{a}{2}$$

C) METHOD OF MULTIPOLE MOMENTS
series expansion of potential about origin or infinity

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\int dq'\, r'\cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \vec{p} = \int dq'\,\vec{r}\,'$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int dq'\, r'^2\frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{4\pi\varepsilon_0 r^5}\int dq'\,\frac{1}{2}\left(3(\vec{r}\cdot\vec{r}\,')^2 - r^2\right) \qquad Q_{xx}+Q_{yy}+Q_{zz} = 0$$

$$= \frac{\frac{1}{2}\vec{r}\cdot\overleftrightarrow{Q}\cdot\vec{r}}{4\pi\varepsilon_0 r^5}$$

$$\overleftrightarrow{Q} = \int dq'\,(3\vec{r}\,'\vec{r}\,' - I r'^2) = \int dq' \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3y'x' & 3y'^2 - r'^2 & 3y'z' \\ 3z'x' & 3z'y' & 3z'^2 - r'^2 \end{pmatrix}$$

$$Q_{ij} = \int dq'\,(3r_i'r_j' - \delta_{ij}r'^2)$$

$$Q^{(\ell)}_{ext} = \frac{A_\ell}{4\pi\varepsilon_0} = \int dq'\,\frac{1}{r'^{\ell+1}}P_\ell(\cos\theta')$$

$$Q^{(\ell)}_{int} = \frac{B_\ell}{4\pi\varepsilon_0} = \int dq'\, r'^\ell\, P_\ell(\cos\theta')$$

* Chapter 4: Dielectric Materials - Dipole

$$\vec{p} = \alpha\vec{E} \qquad \varepsilon_0\chi_e \cong N\alpha$$

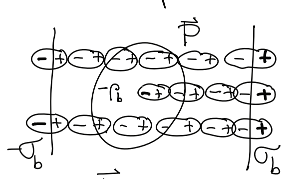
$$\vec{N} = \vec{p}\times\vec{E} \qquad U = -\vec{p}\cdot\vec{E} \qquad \vec{F} = \nabla(\vec{p}\cdot\vec{E})$$

$$\vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{\Delta\vec{p}}{\Delta\tau} = \frac{\Delta n}{\Delta\tau}\vec{p} = N\alpha\vec{E}$$

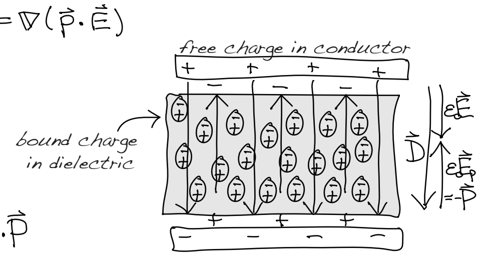
$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}\vec{E} = \underbrace{\varepsilon_0\,\varepsilon_r}_{\varepsilon}\vec{E} = \varepsilon\vec{E}$$

$$\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$$

$$\nabla\cdot\vec{D} = \nabla\cdot\varepsilon_0\vec{E} + \nabla\cdot\vec{P} = \rho - \rho_b = \rho_f$$



$$\sigma_b = \vec{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\vec{P}$$



* Outlook - road to electrodynamic equations

$$\lambda \xrightarrow{d} (V,\vec{A}) \xrightarrow{d} (\vec{E},\vec{B}) \xrightarrow{d} 0$$

$$\varepsilon \,\|\, \mu \qquad \sigma, F$$

$$(\vec{C},U) \xrightarrow{d} (\vec{D},\vec{H}) \xrightarrow{d} (\rho, J) \xrightarrow{d} 0$$

(0) (1) (2) (3) (4)

$$\Phi_D = Q_{encl} \qquad \Phi_B = 0 \qquad -\Box^2(V,\vec{A}) = (\rho/\varepsilon, \mu\vec{J})$$

$$\mathcal{E}_E = -\frac{\partial\Phi_B}{\partial t} \qquad \mathcal{E}_H = I_{encl} + \frac{\partial\Phi_D}{\partial t} \qquad \text{(wave equation)}$$

$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B}) = (\rho\vec{E}+\vec{J}\times\vec{B})d\tau$	Lorentz force
$\partial_t\rho + \nabla\cdot\vec{J} = 0$	Continuity
$\nabla\cdot\vec{D} = \rho \qquad \nabla\times\vec{E} + \partial_t\vec{B} = \vec{0}$	Maxwell electric,
$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{H} - \partial_t\vec{D} = \vec{J}$	magnetic fields
$\vec{D} = \varepsilon\vec{E} \qquad \vec{B} = \mu\vec{H} \qquad \vec{J} = \sigma\vec{E}$	Constitution
$\vec{E} = -\nabla V - \partial_t\vec{A} \qquad \vec{B} = \nabla\times\vec{A}$	Potentials
$V \to V - \partial_t\lambda \qquad \vec{A} \to \vec{A} + \nabla\lambda$	Gauge transform