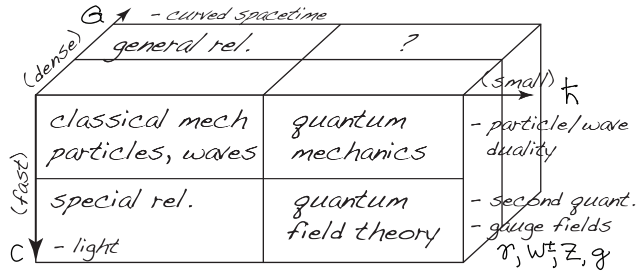
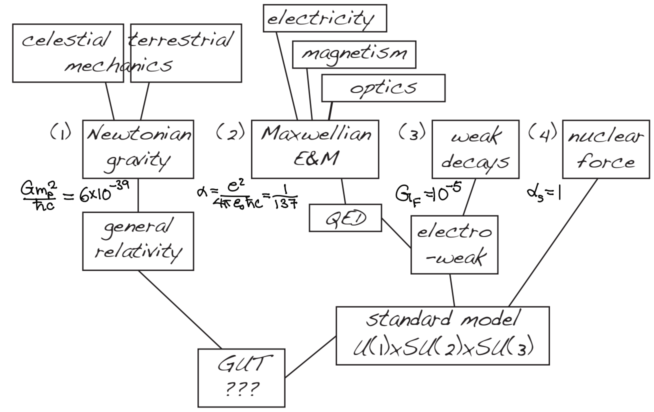
Survey of Electromagnetism

* Realms of Mechanics



~ E&M was second step in unification
~ the stimulus for special relativity
~ the foundation of QED -> standard model

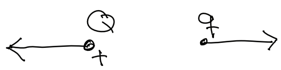
* Unification of Forces



* Electric charge (duFay, Franklin)

~ +,- equal & opposite (QCD: r+g+b=0)
~ $e=1.6\times10^{-19}$ C, quantized ($q_n < 2\times10^{-21}$ e)
~ locally conserved (continuity)

* Electric Force (Coulomb, Cavendish)



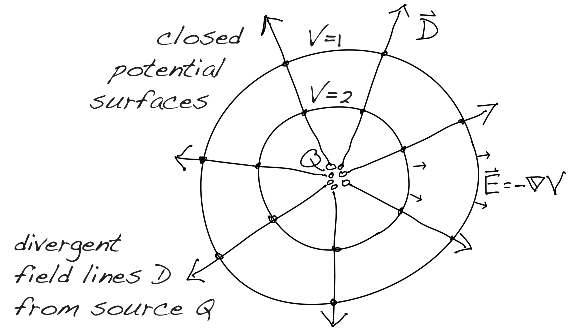
$$\vec{F} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}_{\vec{E}} \cdot q$$

* Electric Field (Faraday)

~ action at a distance vs. locality
field "mediates" or carries force
extends to quantum field theories
~ field is everywhere always $\vec{E}(x,t)$
differentiable, integrable
field lines, equipotentials
~ powerful techniques
for solving complex problems

* Electric potential

$\underset{\text{force}}{\vec{F}} = q\underbrace{\vec{E}}_{\text{field}}$	$\vec{F} = m\underbrace{\vec{g}}_{\text{grav. field}}$
$\underset{\text{energy}}{U} = q\underbrace{Ed}_{\text{potential}}$	$U = m\underbrace{gh}_{\text{"danger"}}$



* Field lines / Flux

~ E is tangent to the field lines
Flux = # of field lines
~ density of the lines = field strength
D is called "electric flux density"
~ note: $\frac{A}{r^2} = \Omega$ independent of distance

$$\boxed{\Phi_D = \int D \cdot d\vec{a}}$$

electric flux
flows from
(+) ⟶ (−)

$$\vec{D} = \varepsilon\vec{E} = \Phi_D/A$$

all flux lines begin at +
and end at − charge

* Equipotential surfaces / Flow

~ no work done to field lines
Equipotentials = surfaces of const energy
~ work is done along field line
Flow = # of potential surfaces crossed

$$\boxed{\mathcal{E}_E \equiv \int \vec{E}\cdot d\vec{\ell}}$$

$$V = -\mathcal{E}_E$$

$$E = -\nabla V$$

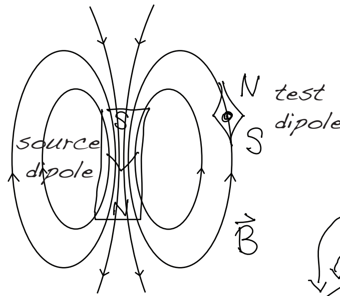
~ potential if flow
is independent of path
~ circulation or EMF in a closed loop

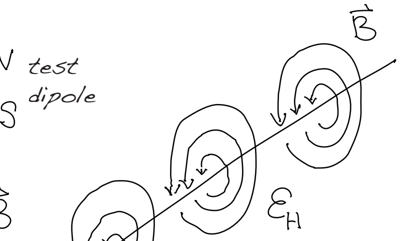
* Magnetic field
- ~ no magnetic charge (monopole)
- ~ field lines must form loops
- ~ permanent magnetic dipoles first discovered

  torque: $\vec{\tau} = \vec{\mu} \times \vec{B}$

  energy: $U = -\vec{\mu} \cdot \vec{B}$

  force: $\vec{F} = \nabla(\vec{\mu} \cdot \vec{B})$

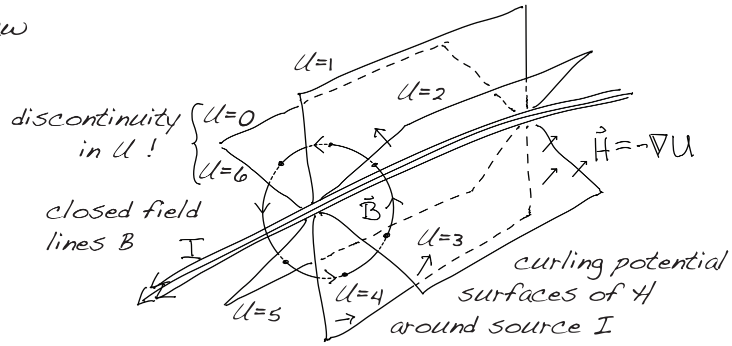




$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$

- ~ electric current shown to generate fields (Oersted, Ampere)
- ~ magnetic dipoles are current loops
- ~ Biot-Savart law - analog of Coulomb law

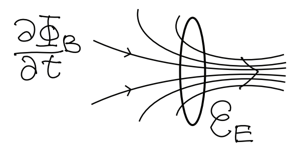
$$\vec{F} = \int I' d\vec{l} \times \underbrace{\frac{\mu_0}{4\pi}\int \frac{I d\vec{l} \times \hat{r}}{r^2}}_{\vec{B}}$$



- ~ B = flux density
- ~ H = field intensity $\quad \vec{B} = \mu\vec{H} = \Phi_B / A$

* Faraday law
- ~ opposite of Orsted's discovery:

  changing magnetic flux induces potential (EMF)
- ~ electric generators, transformers $\quad \mathcal{E}_E = -\frac{\partial \Phi_B}{\partial t}$



* Maxwell equations
- ~ added displacement current - D lines have +/- charge at each end
- ~ changing diplacement current equivalent to moving charge
- ~ derived conservation of charge and restored symmetry in equations
- ~ predicted electromagnetic radiation at the speed of light $\quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$+ \xrightarrow{\Phi_D} -$

$I_d = \frac{\partial \Phi_D}{\partial t}$

Maxwell equations

$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{E} + \partial_t \vec{B} = \vec{0}$

$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{H} - \partial_t \vec{D} = \vec{J}$

Constitutive equations

$\vec{D} = \varepsilon \vec{E} \quad \vec{B} = \mu \vec{H} \quad \vec{J} = \sigma \vec{E}$

$$\boxed{\begin{array}{ll} \Phi_D = Q_{encl} & \Phi_B = 0 \\ \mathcal{E}_E = -\frac{\partial \Phi_B}{\partial t} & \mathcal{E}_H = I_{encl} + \frac{\partial \Phi_D}{\partial t} \end{array}}$$

Lorentz force

$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \int(\rho \vec{E} + \vec{J} \times \vec{B})$

Continuity

$\nabla \cdot \vec{J} + \partial_t \rho = 0$

$$\begin{array}{ccccccccc} & (0) & & (1) & & (2) & & (3) & (4) \\ 0 \to & \lambda & \xrightarrow{d} & (V, \vec{A}) & \xrightarrow{d} & (\vec{E}, \vec{B}) & \xrightarrow{d} & 0 & \\ & & & & & \varepsilon\downarrow \uparrow\mu \searrow^{\sigma} & & & \\ & & & (\,, U) & \xleftarrow{d} & (\vec{D}, \vec{H}) & \xrightarrow{d} & (\rho, \vec{J}) & \xrightarrow{d} 0 \end{array}$$

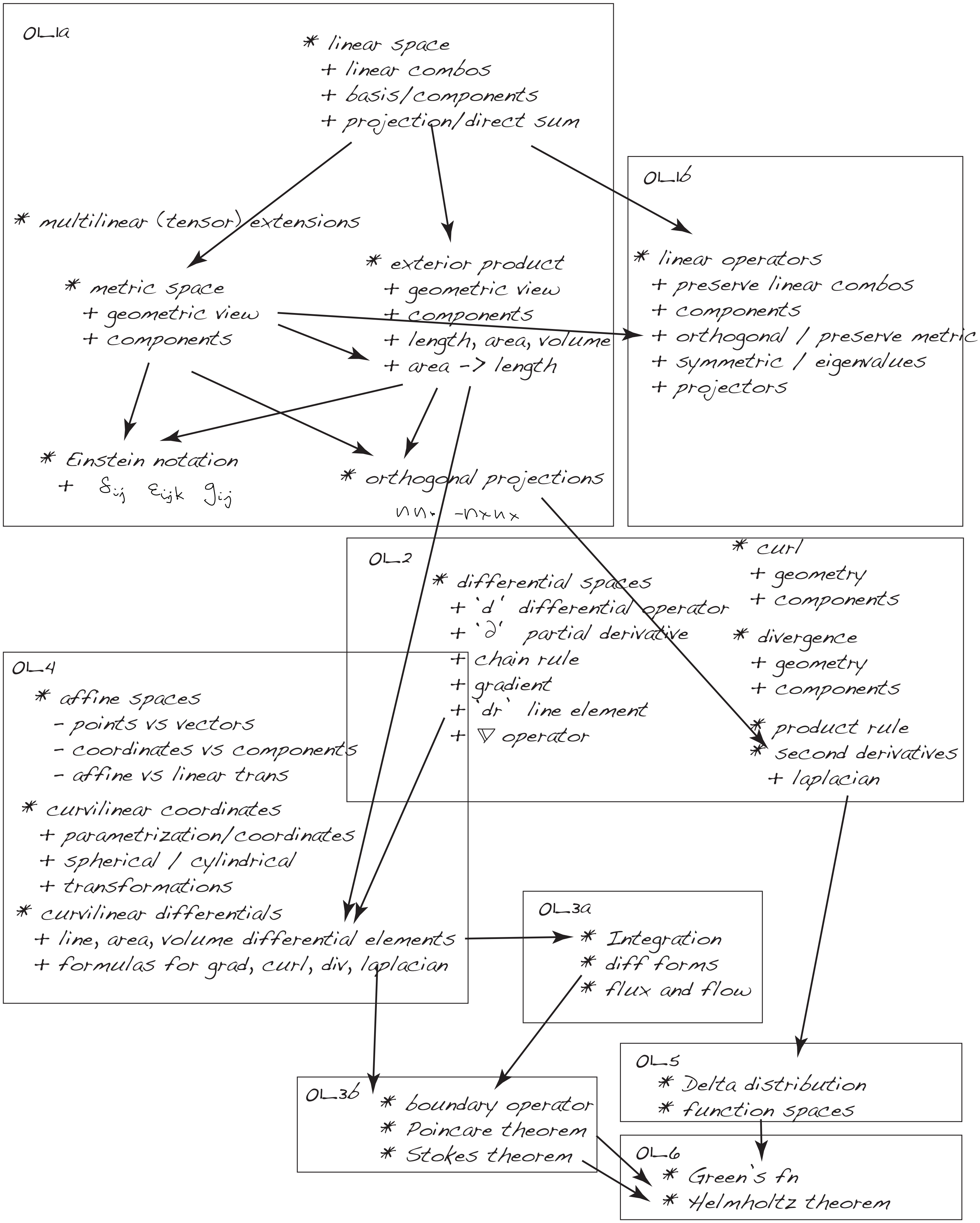
Potentials

$\vec{E} = -\nabla V - \partial_t \vec{A} \quad \vec{B} = \nabla \times \vec{A}$

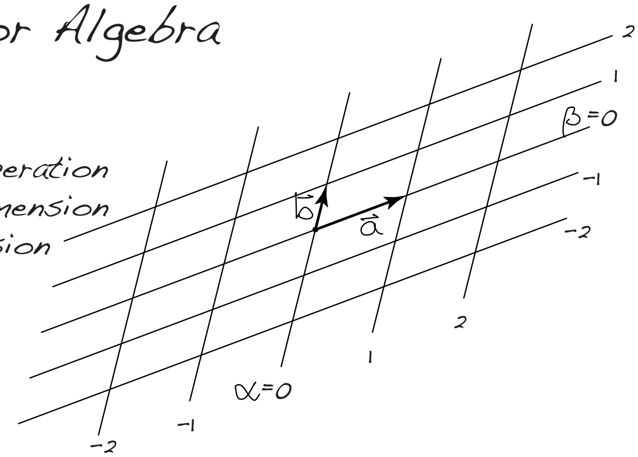
Gauge transformation

$V \to V - \partial_t \lambda \quad \vec{A} \to \vec{A} + \nabla \lambda$

Wave equation $\quad -\Box^2 (V, \vec{A}) = (\rho/\varepsilon, \mu \vec{J})$


OL1a
* linear space
+ linear combos
+ basis/components
+ projection/direct sum
* multilinear (tensor) extensions
* metric space
+ geometric view
+ components
* exterior product
+ geometric view
+ components
+ length, area, volume
+ area -> length
* Einstein notation
+ δij εijk gij
* orthogonal projections
nn· -nxnx
OL1b
* linear operators
+ preserve linear combos
+ components
+ orthogonal / preserve metric
+ symmetric / eigenvalues
+ projectors
OL2
* differential spaces
+ 'd' differential operator
+ '∂' partial derivative
+ chain rule
+ gradient
+ 'dr' line element
+ ∇ operator
* curl
+ geometry
+ components
* divergence
+ geometry
+ components
* product rule
* second derivatives
+ laplacian
OL4
* affine spaces
- points vs vectors
- coordinates vs components
- affine vs linear trans
* curvilinear coordinates
+ parametrization/coordinates
+ spherical / cylindrical
+ transformations
* curvilinear differentials
+ line, area, volume differential elements
+ formulas for grad, curl, div, laplacian
OL3a
* Integration
* diff forms
* flux and flow
OL3b
* boundary operator
* Poincare theorem
* Stokes theorem
OL5
* Delta distribution
* function spaces
OL6
* Green's fn
* Helmholtz theorem

Section 1.1 - Vector Algebra



* Linear spaces

~ linear combination: $(\alpha\vec{u}+\beta\vec{v})$ is the basic operation

~ basis: $(\hat{x},\hat{y},\hat{z}$ or $\vec{a},\vec{b},\vec{c})$ # basis elements = dimension
independence: not collapsed into lower dimension
closure: vectors span the entire space

~ components: $\vec{x} = \vec{a}\alpha+\vec{b}\beta+\vec{c}\gamma = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$
in matrix form: $\vec{x} = \mathbb{b}\,\mathbb{x}$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

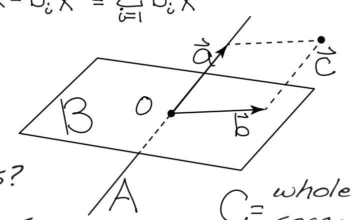
where

$$\vec{a} = \hat{x}a_x+\hat{y}a_y+\hat{z}a_z = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}$$

(usually one upper, one lower index)

~ Einstein notation: implicit summation over repeated indices $\vec{x} = \vec{b}_i x^i \equiv \sum_{i=1}^{3}\vec{b}_i x^i$

~ direct sum: $C = A \oplus B$ add one vector from each independent space to get vector in the product space (not simply union)



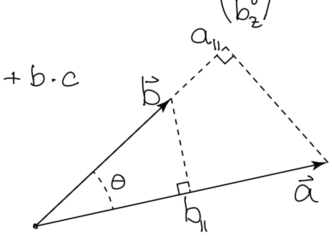
~ projection: the vector $\vec{c}=\vec{a}+\vec{b}$ has a unique decomposition ('coordinates' $(\vec{a},\vec{b})$ in A, B) - relation to basis/components?

~ all other structure is added on as multilinear (tensor) extensions

* Metric (inner, dot) product - distance and angle

$$c = \vec{a}\cdot\vec{b} = ab\cos\theta = a_{\parallel}b = ab_{\parallel} = a_xb_x+a_yb_y+a_zb_z = a_ib^i = (a_x\,a_y\,a_z)\begin{pmatrix}b_x\\ b_y\\ b_z\end{pmatrix}$$

~ properties: 1) scalar valued - what is outer product?
2) bilinear form $a\cdot(b+c) = a\cdot b+a\cdot c$ $(a+b)\cdot c = a\cdot c+b\cdot c$
3) symmetric $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$



~ orthonormality and completeness - two fundamental identities help to calculate components, implicitly in above formulas

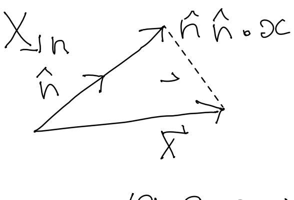
$$\boxed{\begin{gathered}\hat{e}_i\cdot\hat{e}_j = \delta_{ij}\\ \sum_{i=1}^{3}\hat{e}_i\hat{e}_i\cdot = \mathbb{I}\end{gathered}}$$

Kroneker delta: components of the identity matrix $\delta_{ij} = \begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases} \sim \begin{pmatrix}1&0&0\\ 0&1&0\\ 0&0&1\end{pmatrix} = \mathbb{I}$

$\vec{x} = \vec{b}_i x^i \equiv \quad \vec{b}_i x^i$

$$a^i = \vec{a}\cdot\hat{e}^i = a^1\hat{e}_1\cdot\hat{e}^i + a^2\hat{e}_2\cdot\hat{e}^i + a^3\hat{e}_3\cdot\hat{e}^i$$

~ orthogonal projection: a vector $\hat{n}$ divides the space X into $X_{\parallel n}\oplus X_{\perp n}$
geometric view: dot product $\hat{n}\cdot\vec{x}$ is length of $\vec{x}$ along $\hat{n}$
Projection operator: $P_{\parallel} \equiv \hat{n}\hat{n}\cdot$ acts on x: $P_{\parallel}\vec{x} = \vec{x}_{\parallel} = \hat{n}\hat{n}\cdot\vec{x}$



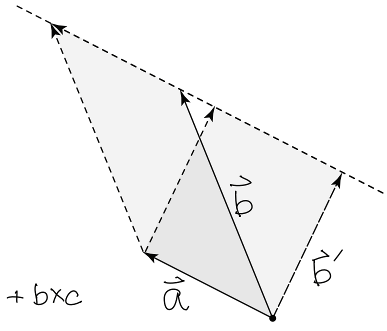
~ generalized metric: for basis vectors which are not orthonormal, collect all nxn dot products into a symmetric matrix (metric tensor) $g = \begin{pmatrix}g_{11}&g_{12}&g_{13}\\ g_{21}&g_{22}&g_{23}\\ g_{31}&g_{32}&g_{33}\end{pmatrix}$

$$g_{ij} = \vec{b}_i\cdot\vec{b}_j \qquad \vec{x}\cdot\vec{y} = x^i\,\vec{b}_i\cdot\vec{b}_j x^j = x^i g_{ij} x^j = (x^1\,x^2\,x^3)\begin{pmatrix}g_{11}&g_{12}&g_{13}\\ g_{21}&g_{22}&g_{23}\\ g_{31}&g_{32}&g_{33}\end{pmatrix}\begin{pmatrix}y^1\\ y^2\\ y^3\end{pmatrix}$$

$$= \mathbb{x}^T\mathbb{b}^T\cdot\mathbb{b}\mathbb{x} = \mathbb{x}^T g\,\mathbb{x} =$$

in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)

Exterior Products – higher-dimensional objects



$\vec{a}\times\vec{b} = \vec{a}\times\vec{b}'$
$\vec{a}\times(\vec{b}-\vec{b}') = \vec{0}$
(parallel)

* cross product (area)

$$\vec{c} = \vec{a}\times\vec{b} = \hat{n}\, ab\sin\theta = \hat{n}\, a_\perp b = \hat{n}\, a b_\perp = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{n}\perp\vec{a}$ and $\hat{n}\perp\vec{b}$ (RH-rule)

~ properties: 1) vector-valued
2) bilinear $\quad a\times(b+c) = a\times b + a\times c \quad (a+b)\times c = a\times c + b\times c$
3) antisymmetric $\quad \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

~ components: $\hat{e}_i \times \hat{e}_j = \varepsilon_{ij}{}^{k}\,\hat{e}_k \qquad \varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ even permutation} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index} \end{cases}$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

Levi-Civita tensor – completely antisymmetric:

$$\vec{x}\times\vec{y} = x^i\,\vec{b}_i\times\vec{b}_j\,y^j = \varepsilon_{ij}{}^{k}\,x^i y^j\,\hat{e}_k$$

~ orthogonal projection: $\hat{n}\times$ projects $\perp$ to $\hat{n}$ and rotates by 90°

$$\hat{x}_\perp = -\hat{n}\times(\hat{n}\times\hat{x}) = P_\perp \vec{x} \qquad P_\perp = -\hat{n}\times\hat{n}\times \qquad \boxed{P_\parallel + P_\perp = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$$

~ where is the metric in $\times$?
vector $\times$ vector = pseudovector
symmetries act more like a 'bivector'
can be defined without metric

* triple product (volume of parallelpiped) – base times height $\quad d = \vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

~ completely antisymmetric – definition of determinant
~ why is the scalar product symmetric / vector product antisymmetric?
~ vector vector $\times$ vector = pseudoscalar (transformation properties)
~ acts more like a 'trivector' (volume element)
~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)
~ extended vector space with basis elements from objects of each dimension
~ pseudo-vectors, scalar separated from normal vectors, scalar

magnitude,	length,	area,	volume
scalar,	vectors,	bivectors,	trivector
1,	$\hat{x},\hat{y},\hat{z}$,	$\hat{x}\hat{y},\hat{y}\hat{z},\hat{z}\hat{x}$,	$\hat{x}\hat{y}\hat{z}$

~ what about higher-dimensional spaces (like space-time)?
can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements
most important example: BAC-CAB rule (HW1: relation to projectors)

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$\varepsilon^i{}_{jk}A^j(\varepsilon^k{}_{mn}B^m C^n) = (\delta^i_m\delta_{jn} - \delta^i_n\delta_{jm})A^j B^m C^n = B^i(A^j C_j) - C^i(A^j B_j)$$

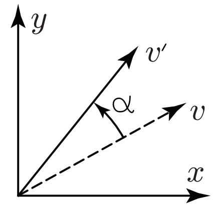
Section 1.1.5 - Linear Operators

* Linear Transformation
  ~ function which preserves linear combinations
  ~ determined by action on basis vectors (egg-crate)
  ~ rows of matrix are the image of basis vectors
  ~ determinant = expansion volume (triple product)
  ~ multilinear (2 sets of bases) - a tensor

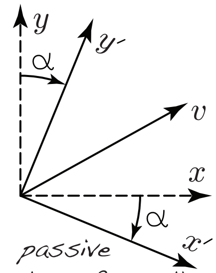
$$M(\alpha\vec{a}+\beta\vec{b}) = \alpha M(\vec{a}) + \beta M(\vec{b})$$

$$M\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{M\begin{pmatrix}1\\0\end{pmatrix}}_{\vec{m}_1} x + \underbrace{M\begin{pmatrix}0\\1\end{pmatrix}}_{\vec{m}_2} y = \left(\begin{array}{c|c} m_{1x} & m_{2x} \\ m_{1y} & m_{2y}\end{array}\right)\begin{pmatrix}x\\y\end{pmatrix}$$

* Change of coordinates
  ~ two ways of thinking about transformations
    both yield the same transformed components
  ~ active: basis fixed, physically rotate vector
  ~ passive: vector fixed, physically rotate basis



active transformation



passive transformation

* Transformation matrix (active) - basis vs. components

$$(\vec{a}\,\vec{b}\,\vec{c}) = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z\end{pmatrix} \qquad \vec{X} = (\vec{a}\,\vec{b}\,\vec{c})\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = (\hat{x}\,\hat{y}\,\hat{z})\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\vec{\mathbb{e}}' = \vec{\mathbb{e}}\,\mathcal{R} \qquad \vec{x} = \overbrace{\vec{\mathbb{e}}'\,\mathbb{X}'} = \overbrace{\vec{\mathbb{e}}\,\mathcal{R}}\underbrace{\mathcal{R}\mathbb{X}'} = \vec{\mathbb{e}}\,\underbrace{\mathbb{X}} = \vec{x} \qquad \mathbb{X} = \mathcal{R}\ \mathbb{X}'$$

$$\boxed{\begin{aligned}\vec{\mathbb{e}}' &= \vec{\mathbb{e}}\,\mathcal{R}\\ \mathbb{X}' &= \mathcal{R}^{-1}\mathbb{X}\end{aligned}}$$

* Orthogonal transformations
  ~ R is orthogonal if it 'preserves the metric' (has the same form before and after)

$$\vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}} = \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}\cdot(\hat{x}\ \hat{y}) = \begin{pmatrix}\hat{x}\cdot\hat{x} & \hat{x}\cdot\hat{y}\\ \hat{y}\cdot\hat{x} & \hat{y}\cdot\hat{y}\end{pmatrix} = \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = g \qquad \vec{\mathbb{e}}'^T\cdot\vec{\mathbb{e}}' = \begin{pmatrix}\vec{a}\\ \vec{b}\end{pmatrix}\cdot(\vec{a}\ \vec{b}) = \begin{pmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{pmatrix} = g'$$

$$\vec{\mathbb{e}}' = \vec{\mathbb{e}}\,\mathcal{R} \qquad \vec{\mathbb{e}}'^T\cdot\vec{\mathbb{e}}' = \mathcal{R}^T\vec{\mathbb{e}}^T\cdot\vec{\mathbb{e}}\,\mathcal{R} = \mathcal{R}^T g\,\mathcal{R} = g' \qquad g = g' \qquad \boxed{\mathcal{R}^T g\,\mathcal{R} = g}$$

  ~ equivlent definition in terms of components:

$$\vec{X}\cdot\vec{X} = \overbrace{\mathbb{X}^T} g\,\overbrace{\mathbb{X}} = \overbrace{\mathbb{X}'^T\mathcal{R}^T}\underbrace{g\,\overbrace{\mathcal{R}\mathbb{X}'}} = \mathbb{X}'^T g'\,\mathbb{X}'$$

(metric invariant under rotations if $g = g'$)

  ~ starting with an orthonormal basis: $g = I \quad g_{ij} = \delta_{ij} \qquad \mathcal{R}^T\mathcal{R} = I \qquad \mathcal{R}^{-1} = \mathcal{R}^T$

* Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition
  ~ recall complex numbers $u = \rho + i\phi \qquad \rho^* = \rho \qquad (i\phi)^* = -i\phi$

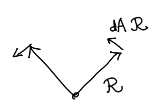
$$e^u = e^{\rho+i\phi} = r e^{i\phi} \qquad |e^{i\phi}|^2 = e^{-i\phi}e^{i\phi} = e^{i0} = 1$$

  ~ similar behaviour of symmetric / antisymmetric matrices

M artibrary matrix
T symmetric
A antisymmetric

S symmetric
R orthogonal

$$M = \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & (b+c)/2\\ (b+c)/2 & d\end{pmatrix} + \begin{pmatrix} 0 & (b-c)/2\\ (c-b)/2 & 0\end{pmatrix} = T + A$$



$$e^M = 1 + M + \tfrac{1}{2!}M^2 + \tfrac{1}{3!}M^3 + \dots \quad = e^{T+A} \neq e^T e^A$$

$$S = e^T = e^{VWV^{-1}} = V e^W V^{-1} \qquad R = e^A \qquad R^T R = (e^A)^T e^A = e^{A^T + A} = e^0 = I$$

$$\det\begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots\end{pmatrix} = e^{\lambda_1}\cdot e^{\lambda_2}\dots = e^{\lambda_1+\lambda_2+\dots} = e^{\operatorname{tr}\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}} \qquad \det e^A = e^{\operatorname{tr} A} = e^0 = 1$$

Eigenparaphernalia

* illustration of symmetric matrix S with eigenvectors v, eigenvalues λ

$$S\,v = \lambda v$$

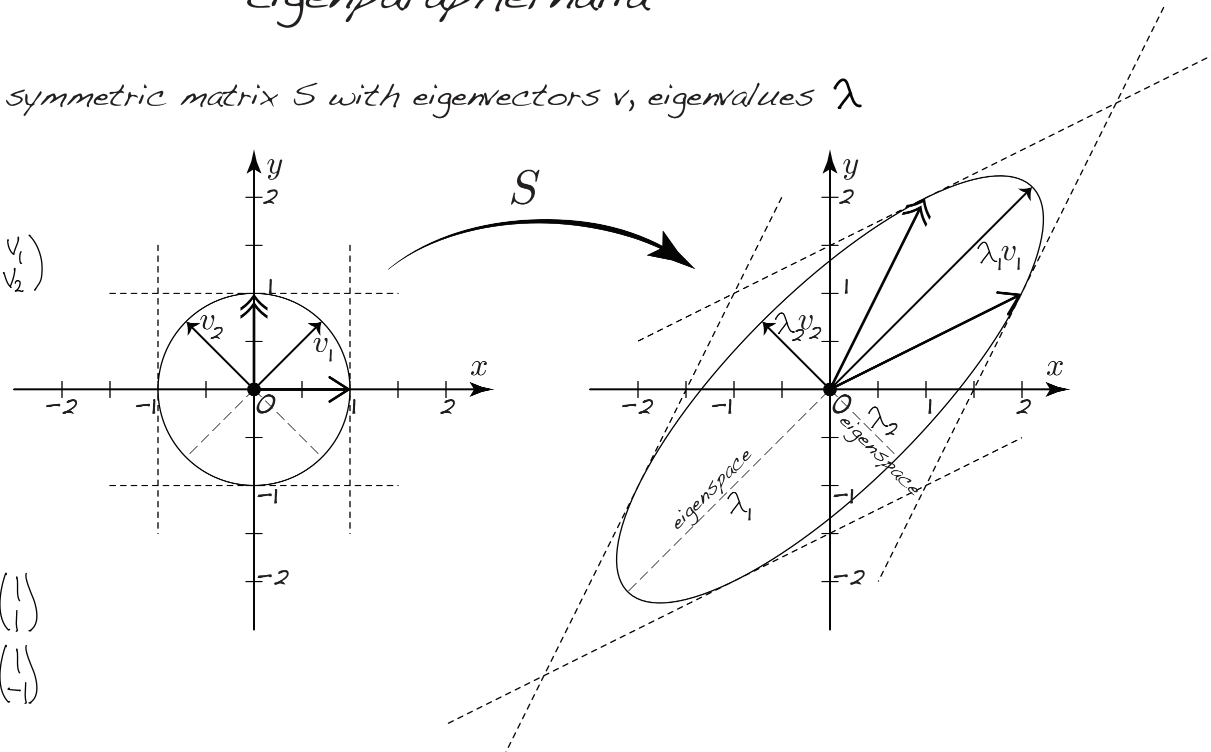
$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \lambda\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 1\begin{pmatrix}1\\-1\end{pmatrix}$$



* similarity transform - change of basis (to diagonalize A)

$$S\,(v_1 v_2 \ldots) = (\vec{v}_1 \vec{v}_2 \ldots)\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix} \qquad S\,V = V\,W\,V^{-1} = V\,W\,V^{T}$$

* a symmetric matrix has real eigenvalues

$$S\,v = \lambda v \qquad v^{*T} S\, v = \lambda\, v^{*T} v$$

$$v^{*T} S = v^{*T} \lambda^{*} \qquad v^{*T} S\, v = \lambda^{*}\, v^{*T} v \qquad \lambda = \lambda^{*}$$

~ what about a antisymmetric/orthogonal matrix?

* eigenvectors of a symmetric matrix with distinct eigenvalues are orthogonal

$$v^T S = (S^T v)^T = (S v)^T = (\lambda v)^T = v^T \lambda$$

$$\lambda_1\, v_1 \cdot v_2 = (v_1^T S)\, v_2 = v_1^T (S\, v_2) = v_1 \cdot v_2\, \lambda$$

$$v_1 \cdot v_2\, (\lambda_1 - \lambda_2) = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \text{ then } v_1 \cdot v_2 = 0.$$

* singular value decomposition (SVD)

~ transformation from one orthogonal basis to another

$$M = R\,S = \underbrace{R\,V}\,W\,V^T = U\,W\,V^T$$

~ extremely useful in numerical routines

M arbitrary matrix
R orthogonal
S symmetric
W diagonal matrix
V orthogonal (domain)
U orthogonal (range)

Section 1.2 - Differential Calculus

* differential operator

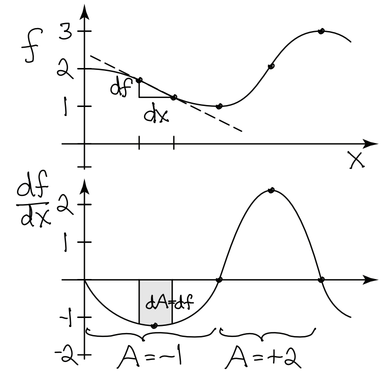
~ ex. $u = x^2 \qquad du = dx^2 = 2x\,dx$ $\qquad d \equiv \lim_{\Delta\to 0} \Delta \approx 0$

or $d(\sin x^2) = \cos(x^2)\, d\, x^2 = \cos x^2 \cdot 2x \cdot dx$

~ df and dx connected - refer to the same two endpoints

~ made finite by taking ratios (derivative or chain rule) or inifinite sum = integral (Fundamental Thereom of calculus)

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \int_a^b \frac{df}{dx}\,dx = \int_a^b df = f\Big|_a^b$$

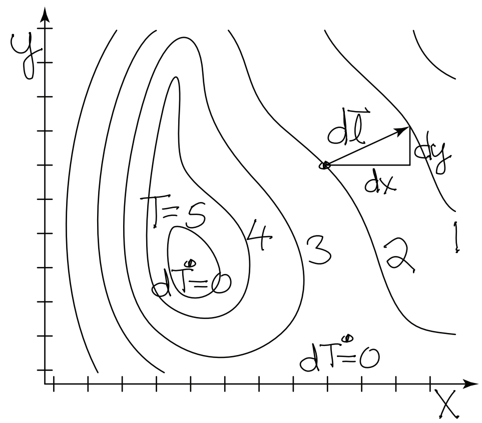


* scalar and vector fields - functions of position ($\vec{r}$)

~ "field of corn" has a corn stalk at each point in the field

~ scalar fields represented by level curves (2d) or surfaces (3d)

~ vector fields represented by arrows, field lines, or equipotentials



* partial derivative & chain rule

~ signifies one varying variable AND other fixed variables

~ notation determined by denominator; numerator along for the ride

~ total variation split into sum of variations in each direction

$$\frac{\partial u}{\partial x} \quad \left(\frac{\partial u}{\partial x}\right)_{y,z} \quad \partial_x u \quad u_{,x} \qquad\qquad \frac{\cdots}{\cdots} = \frac{dx}{\cdots}\frac{\cdots}{\partial x} + \frac{dy}{\cdots}\frac{\cdots}{\partial y} + \frac{dz}{\cdots}\frac{\cdots}{\partial z}$$

* vector differential - gradient

~ differential operator , del operator

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla} T \cdot \underbrace{(dx, dy, dz)}_{d\vec{\ell}}$$

$$\boxed{\begin{aligned} d &= dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} = d\vec{r}\cdot\nabla \\ \nabla &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \frac{d}{d\vec{r}} \\ d\vec{\ell} &= \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz = d\vec{r} \end{aligned}}$$

~ differential line element: $\cdot d\vec{\ell}$ and $\frac{\ }{d\vec{\ell}}$ transforms between $\hat{x},\hat{y},\hat{z} \leftrightarrow dx, dy, dz$ and $d \leftrightarrow \nabla$

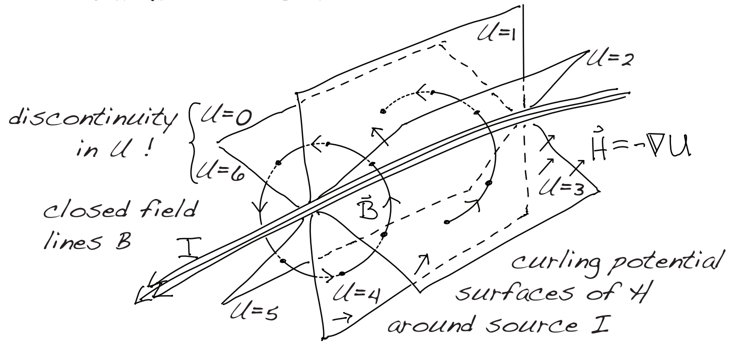
~ example: $d\,x^2y = 2xy\,dx + x^2dy = (2xy, x^2)\cdot(dx, dy)$

~ example: let $z = f(x,y)$ be the graph of a surface. What direction does ∇f point?

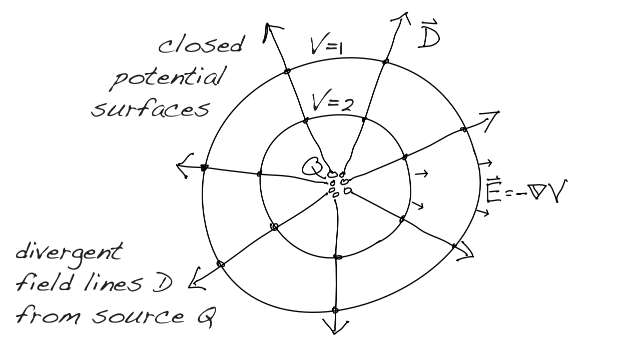
now let $g = z - f(x,y)$ so that $g = 0$ on the surface of the graph

then $\nabla g = (-f_x, -f_y, 1)$ is normal to the surface

* illustration of curl



* illustration of divergence



Higher Dimensional Derivatives

* curl - circular flow of a vector field

$$\nabla\times\vec{V} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ V_x & V_y & V_z\end{vmatrix} = \begin{matrix}\hat{x}\,(V_{z,y}-V_{y,z})\\ +\hat{y}\,(V_{x,z}-V_{z,x})\\ +\hat{z}\,(V_{y,x}-V_{x,y})\end{matrix}$$

* divergence - radial flow of a vector field

$$\nabla\cdot\vec{V} = (\partial_x\ \partial_y\ \partial_z)\begin{pmatrix}V_x\\ V_y\\ V_z\end{pmatrix} = V_{x,x}+V_{y,y}+V_{z,z}$$

* product rules

~ how many are there?

~ examples of proofs

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$\vec{A}\times(\nabla\times\vec{B}) = \nabla(\vec{A}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\nabla)$$

$$\nabla\times(\vec{A}\times B) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

$$\nabla(fg) = \nabla f\cdot g + f\cdot\nabla g$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\times(f\vec{A}) = \nabla f\times\vec{A} + f(\nabla\times\vec{A})$$

$$\nabla\times(\vec{A}\times\vec{B}) = (B\cdot\nabla)A - B(\nabla\cdot A) - (\vec{B}\leftrightarrow\vec{A})$$

$$\nabla\cdot(f\vec{A}) = \nabla f\cdot\vec{A} + f\,\nabla\cdot\vec{A}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})$$

* second derivatives - there is really only ONE! (the Laplacian) $\boxed{\nabla^2 \equiv \nabla\cdot\nabla \equiv \partial_x^2+\partial_y^2+\partial_z^2}$

1) $\nabla\cdot(\nabla T) = \nabla^2 T$ ~ eg: $\nabla^2 T = 0$ no net curvature - stretched elastic band

$(\nabla\cdot\nabla)\vec{v} = \nabla^2\vec{v}$ ~ defined component-wise on v_x, v_y, v_z (only cartesian coords)

3), 5) $\nabla^2 = \nabla_\parallel^2 + \nabla_\perp^2$ ~ longitudinal / transverse projections $\nabla(\nabla\cdot\vec{v}) \equiv \nabla_\parallel^2\vec{v}$

$= \nabla(\nabla\cdot \ -\nabla\times\nabla\times$ $\quad\vec{k}\cdot\vec{k} = \vec{k}\vec{k}\cdot\ - \vec{k}\times(\vec{k}\times$ $\quad -\nabla\times\nabla\times\vec{v} \equiv -\nabla_\perp^2\vec{v}$

2), 4) $\nabla\times\nabla = 0$ $\quad\nabla\times(\nabla T) = 0\quad \nabla\cdot(\nabla\times\vec{v}) = 0$

$$\nabla\times\nabla = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ \partial_x & \partial_y & \partial_z\end{vmatrix} = \begin{matrix}\hat{x}\,(\partial_y\partial_z-\partial_z\partial_y)\\ +\hat{y}\,(\partial_z\partial_x-\partial_x\partial_z)\\ +\hat{z}\,(\partial_x\partial_y-\partial_y\partial_x)\end{matrix}$$

~ equality of mixed partials ($d^2=0$)

* unified approach to all higher-order derivatives with differential operator

1) $d^2 = 0$ 2) $dx^2 = 0$ 3) $dx\,dy = -dy\,dx$ + differential (line, area, volume) elements

~ Gradient

$$df = f_{,x}dx + f_{,y}dy + f_{,z}dz = \nabla f\cdot d\vec{l} \qquad d\vec{l} = (dx, dy, dz) = d\vec{r}$$

~ Curl

$$d(\vec{A}\cdot d\vec{l}) = d(A_x dx + A_y dy + A_z dz)$$
$$= A_{x,x}\cancel{dx\,dx} + A_{x,y}\,dy\,dx + A_{x,z}\,dz\,dx$$
$$+ A_{y,x}\,dx\,dy + A_{y,y}\cancel{dy\,dy} + A_{y,z}\,dz\,dy$$
$$+ A_{z,x}\,dx\,dz + A_{z,y}\,dy\,dz + A_{z,z}\cancel{dz\,dz}$$
$$= (A_{z,y}-A_{y,z})dy\,dz + (A_{x,z}-A_{z,x})dz\,dx + (A_{y,x}-A_{x,y})dx\,dy$$

$$d(\vec{A}\cdot d\vec{l}) = (\nabla\times\vec{A})\cdot d\vec{a}$$

$$d\vec{a} = (dy\,dz, dz\,dx, dx\,dy) = \tfrac{1}{2}d\vec{l}\times d\vec{l} = d^2\vec{r}$$

~ Divergence

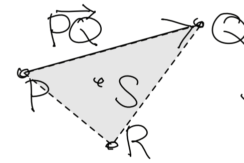
$$d(\vec{B}\cdot d\vec{a}) = d(B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy)$$
$$= B_{x,x}\,dx\,dy\,dz + B_{x,y}\cancel{dy\,dy\,dz} + B_{x,z}\cancel{dz\,dy\,dz}$$
$$+ B_{y,x}\cancel{dx\,dz\,dx} + B_{y,y}\,dy\,dz\,dx + B_{y,z}\cancel{dz\,dz\,dx}$$
$$+ B_{z,x}\cancel{dx\,dx\,dy} + B_{z,y}\cancel{dy\,dx\,dy} + B_{z,z}\,dz\,dx\,dy.$$
$$= (B_{x,x}+B_{y,y}+B_{z,z})\,dx\,dy\,dz$$

$$d(\vec{B}\cdot d\vec{a}) = \nabla\cdot B\cdot d\tau \qquad d\tau = \tfrac{1}{6}d\vec{l}\cdot d\vec{l}\times d\vec{l} = d^3\vec{r}$$

$$\boxed{\nabla f = \frac{df}{d\vec{l}} = \frac{df}{d\vec{r}}} \qquad \boxed{\nabla\times\vec{A} = \frac{d(\vec{A}\cdot d\vec{l})}{d\vec{a}} = \frac{d(d\vec{r}\cdot\vec{A})}{d^2\vec{r}}} \qquad \boxed{\nabla\cdot\vec{B} = \frac{d(\vec{B}\cdot d\vec{a})}{d\tau} = \frac{d(d^2\vec{r}\cdot\vec{B})}{d^3\vec{r}}}$$

Section 1.4 - Affine Spaces

* Affine Space - linear space of points



$$S=\alpha P+\beta Q+\gamma R \qquad \alpha+\beta+\gamma=1$$

POINTS vs VECTORS

~ operations $Q-P=\vec{V}$, $P+\vec{V}=Q$ $\qquad \vec{W}=\alpha\vec{u}+\beta\vec{v}$

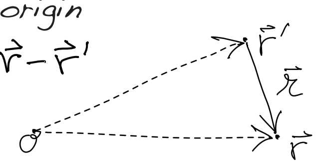
~ points are invariant under translation of the origin

~ can treat points as vectors from the origin to the point
cumbersome picture: many meaninglyess arrows from meaningless origin

position vector: field point $\vec{r}=(x,y,z)$ source pt $\vec{r}'=(x',y',z')$

displacement vector: $\vec{\mathscr{r}}\equiv\vec{r}-\vec{r}'$

differential: $d\vec{l}=\frac{\partial\vec{r}}{\partial q}dq=\vec{b}\,dq$



~ the only operation on points is the weighted average
weight $w=0$ for vectors and $w=1$ for points

~ transformation: affine vs linear

$$\begin{pmatrix} R & \vec{E} \\ 000 & 1\end{pmatrix}\begin{pmatrix}\vec{r}\\ 1\end{pmatrix}=\begin{pmatrix}R\vec{r}+\vec{E}\\ 1\end{pmatrix}$$

$$\begin{pmatrix} R & \vec{E} \\ 000 & 1\end{pmatrix}\begin{pmatrix}\vec{v}\\ 0\end{pmatrix}=\begin{pmatrix}R\vec{v}\\ 0\end{pmatrix}$$

~ decomposition: coordinates vs components
- they appear the same for cartesian systems!
- coordinates are scalar fields $q^i(\vec{r})$

* Rectangular, Cylindrical and Spherical coordinate transformations

~ math: 2-d -> N-d physics: 3d + azimuthal symmetry

~ singularities on z-axis ( ) and origin

$$(\hat{s},\hat{\phi},\hat{z})=(\hat{x},\hat{y},\hat{z})\underbrace{\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}}_{R_{\hat{z}}(\phi)}$$

rect. cyl. sph.

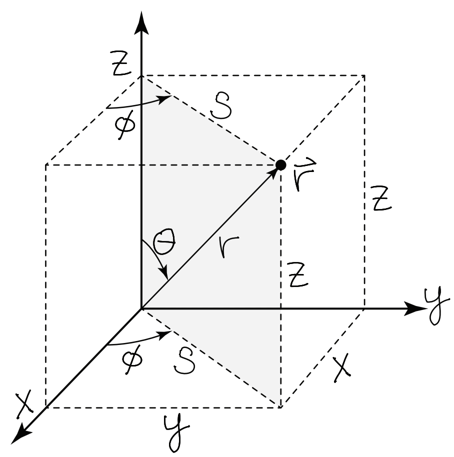
$$x=s\cdot\cos\phi=r\cdot\sin\theta\cdot\cos\phi$$
$$y=s\cdot\sin\phi=r\cdot\sin\theta\cdot\sin\phi$$
$$z=z\qquad=r\cdot\cos\theta$$

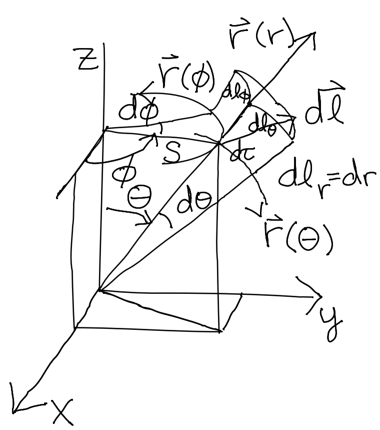
$$(\hat{r}\,\hat{\theta}\,\hat{\phi})=(\hat{s}\,\hat{\phi}\,\hat{z})\underbrace{\begin{pmatrix}\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & -\sin\theta & 0\end{pmatrix}}_{R_{\hat{\phi}}(\theta)}=(\hat{x}\,\hat{y}\,\hat{z})\,R_{\hat{z}}(\phi)\cdot R_{\hat{\phi}}(\theta)$$

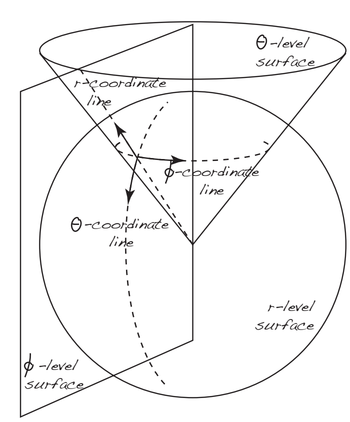
$$d\vec{l}_{rec}=\hat{x}\,dx+\hat{y}\,dy+\hat{z}\,dz$$
$$d\vec{l}_{cyl}=\hat{s}\,ds+\hat{\phi}\,s\,d\phi+z\,dz$$
$$d\vec{l}_{sph}=\hat{r}\underbrace{dr}_{dl_r}+\hat{\theta}\underbrace{r\,d\theta}_{dl_\theta}+\hat{\phi}\underbrace{r\sin\theta\,d\phi}_{dl_\phi}$$

$$d\vec{a}_{rec}=\hat{x}\,dy\,dz+\hat{y}\,dz\,dx+\hat{z}\,dx\,dy$$
$$d\vec{a}_{cyl}=\hat{s}\,s\,d\phi\,dz+\hat{\phi}\,dz\,ds+\hat{z}\,ds\,s\,d\phi$$
$$d\vec{a}_{sph}=\hat{r}\,r\,d\theta\,r\sin\theta\,d\phi+\hat{\theta}\,r\sin\theta\,d\phi\,dr+\hat{\phi}\,dr\,r\,d\theta$$

$$d\tau_{rec}=dx\,dy\,dz$$
$$d\tau_{cyl}=ds\cdot s\,d\phi\cdot dz$$
$$d\tau_{sph}=dr\cdot r\,d\theta\cdot r\sin\theta\,d\phi=r^2dr\,d\Omega$$







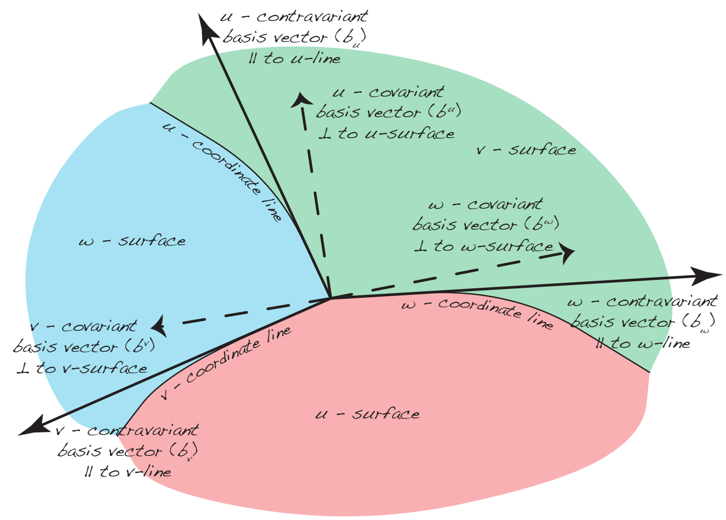
Curvilinear Coordinates

* coordinate surfaces and lines

~ each coordinate is a scalar field $q(\vec{r})$

~ coordinate surfaces: constant q^i

~ coordinate lines: constant q^j, q^k



* coordinate basis vectors

$q^i \sim \{u, v, w\}$ ~ generalized coordinates

$\vec{b}_i = \left(\frac{\partial \vec{r}}{\partial q^i}\right)_{q^j, q^k} \sim \{\hat{u}f, \hat{v}g, \hat{w}h\}$ ~ contravariant basis

$\vec{b}^i = \nabla q^i \sim \{\hat{u}/f, \hat{v}/g, \hat{w}/h\}$ ~ covariant basis

$h_i = |\vec{b}_i| \sim \{f, g, h\}$ ~ scale factor

$\hat{e}_i = \vec{b}_i / h_i \sim \{\hat{u}, \hat{v}, \hat{w}\}$ ~ unit vector

$g_{ij} = \vec{b}_i \cdot \vec{b}_j \sim \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$ ~ metric (dot product)

$\vec{\Gamma}_{ij} = \frac{\partial \vec{b}_j}{\partial q^i} = \vec{b}_k \Gamma^k_{ij}$ ~ Christoffel symbols - derivative of basis vectors

* differential elements

$$d\vec{l} = \frac{\partial \vec{r}}{\partial q^1} dq^1 + \frac{\partial \vec{r}}{\partial q^2} dq^2 + \frac{\partial \vec{r}}{\partial q^3} dq^3 = \vec{b}_i \, dq^i$$

$$= \hat{e}_1 \underbrace{h_1 dq^1}_{dl_1} + \hat{e}_2 \underbrace{h_2 dq^2}_{dl_2} + \hat{e}_3 \underbrace{h_3 dq^3}_{dl_3}$$

$$d\vec{a} = \tfrac{1}{2} d\vec{l} \times d\vec{l} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \\ h_1 dq^1 & h_2 dq^2 & h_3 dq^3 \end{vmatrix}$$

$$= \hat{e}_1 \cdot h_2 dq^2 \cdot h_3 dq^3 + \hat{e}_2 \cdot h_3 dq^3 \cdot h_1 dq^1 + \hat{e}_3 \cdot h_1 dq^1 \cdot h_2 dq^2$$

$$d\tau = \tfrac{1}{2} d\vec{l} \times d\vec{a} = \tfrac{1}{2} d\vec{l} \cdot d\vec{l} \times d\vec{l} = h_1 dq^1 \cdot h_2 dq^2 \cdot h_3 dq^3$$

* example ($c_\phi \equiv \cos\phi$)

$x = s \qquad dx = c_\phi \, ds - s \, s_\phi \, d\phi$

$y = s \, s_\phi \qquad dy = s_\phi \, ds + s c_\phi \, d\phi$

$$d\vec{l} = \hat{x} dx + \hat{y} dy = (\hat{x} c_\phi + \hat{y} s_\phi) ds + (\hat{x} s_\phi - \hat{y} c_\phi) s \, d\phi$$

$$= \hat{s} \, ds + \hat{\phi} \, s \, d\phi \qquad (\hat{s} \;\; \hat{\phi}) = (\hat{x} \;\; \hat{y}) \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}$$

$s^2 = x^2 + y^2 \qquad 2s\,ds = 2x\,dx + 2y\,dy$

$y = x \tan\phi \qquad dy = dx \tan\phi + x \sec^2\phi \, d\phi$

$$d\phi = \frac{-y}{s^2} dx + \frac{x}{s^2} dy$$

$$\nabla s = \frac{x}{s} \hat{x} + \frac{y}{s} \hat{y} = c_\phi \hat{x} + s_\phi \hat{y} = \hat{s}$$

$$\nabla \phi = \frac{-y}{s^2} \hat{x} + \frac{x}{s^2} \hat{y} = \frac{-s_\phi \hat{x} + c_\phi \hat{y}}{s} = \frac{\hat{\phi}}{s}$$

* formulas for vector derivatives in curvilinear coordinates

$$df = \frac{\partial f}{\partial q^i} dq^i = \frac{\partial f}{h_i \partial q^i} \cdot h_i dq^i = \nabla f \cdot d\vec{l}$$

$$\nabla f = \frac{df}{d\vec{r}} = \frac{\hat{e}_i}{h_i} \frac{\partial}{\partial q^i} f$$

$$d(\vec{A} \cdot d\vec{l}) = d\,(A_k h_k dq^k) = \frac{\partial}{\partial q^j} (h_k A_k) \, dq^j dq^k$$

$$= \varepsilon_{ijk} \frac{\partial (h_k A_k)}{h_j h_k \, \partial q^k} da_i = (\nabla \times \vec{A}) \cdot d\vec{a}$$

$$\nabla \times \vec{A} = \frac{d(\vec{A} \cdot d\vec{l})}{d^2 \vec{r}} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial q^1 & \partial/\partial q^2 & \partial/\partial q^3 \\ h_1 A^1 & h_2 A^2 & h_3 A^3 \end{vmatrix}$$

$$d(\vec{B} \cdot d\vec{a}) = d\,(B_i h_j dq^j h_k dq^k) = \frac{\partial}{\partial q^i} (h_j h_k B_i) \, dq^i dq^j dq^k$$

$$= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q} \frac{\partial (h_j h_k B_i)}{\partial q^i} d\tau = \nabla \cdot \vec{B} \, d\tau$$

$$\nabla \cdot \vec{B} = \frac{d(\vec{B} \cdot d^2\vec{r})}{d^3 \vec{r}} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i} (h_j h_k B_i)$$

i, j, k cyclic

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q_i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q_i} f$$

this formula does not work for $\nabla^2 \vec{B}$

instead, use: $\nabla^2 = \nabla \nabla \cdot \; - \nabla \times \nabla \times$

Section 1.3 - Integration

* different types of integration in vector calculus

1-dim: $\omega^{(1)} = \lambda\, dl,\ \varphi\, d\vec{l},\ \vec{A}\, dl,\ \vec{A}\cdot d\vec{l},\ \vec{A}\times d\vec{l}$

2-dim: $\omega^{(2)} = \sigma\, da,\ \sigma\, d\vec{a},\ \vec{B}\, da\ \ \vec{B}\cdot d\vec{a},\ \vec{B}\times d\vec{a}$

3-dim: $\omega^{(3)} = \rho\, d\tau,\quad \vec{F}\, d\tau$

Flow: $\mathcal{E}_A = \int \tilde{A} = \int \vec{A}\cdot d\vec{l}$

Flux: $\Phi_B = \int \tilde{B} = \int \vec{B}\cdot d\vec{a}$

Substance: $Q_\rho = \int \tilde{\rho} = \int \rho\, d\tau$

~ "differential forms" are the things after the all have a 'd' somewhere inside

$d\vec{l}_{rec} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$d\vec{a}_{rec} = \hat{x}\,dy\,dz + \hat{y}\,dz\,dx + \hat{z}\,dx\,dy$

$d\tau_{rec} = dx\,dy\,dz$

~ often $d\vec{l}, d\vec{a}, d\tau$ are burried inside of another 'd'

current element $dq \equiv q_i^{(0)},\ \lambda\, dl^{(1)},\ \sigma\, da^{(2)},\ \rho\, d\tau^{(3)}$

charge element $d\vec{q} \equiv \vec{v} q_i,\ I\, d\vec{l},\ \vec{K}\, da,\ \vec{J}\, d\tau$

~ two types of regions:

over the region R: $\int_R \omega$ (open region)

over the boundary ∂R of R: $\oint_{\partial R} \omega$ (closed region)

* recipe for ALL types of integration

	coordinates on path/surface/volume	boundary of coordinates
1-d	$\mathcal{P}$: $\vec{r}(t)$	$\int_{s=a}^{b} \int_{t=t_0(s)}^{t_1(s)}$
2-d	$\mathcal{S}$ $\vec{r}(s,t)$	
3-d	$\mathcal{V}$ $\vec{r}(s,t,u)$	

a) Parametrize the region

~ parametric vs relations equations of a region

~ boundaries translate to endpoints on integrals

b) Pull back the paramters

~ x,y,z become functions of s,t,u

~ differentials: dx,dy,dz become ds,dt,du

~ reduce using the chain rule

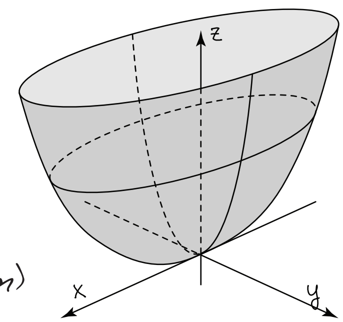
$$d\vec{l} = \frac{d\vec{r}}{dt}dt \qquad d\vec{a} = \frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\,ds\,dt \qquad d\tau = \frac{\partial\vec{r}}{\partial s}\cdot\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u}\,ds\,dt\,du$$

$$x = x(t) \quad dx = x'\,dt \qquad y = y(t) \quad dy = y'\,dt \qquad z = z(t) \quad dz = z'\,dt$$

$$\int_R \vec{A}\cdot d\vec{l} = \int_{\vec{x}(t)} A_x(x,y,z)\,dx + A_y(x,y,z)\,dy + A_z(x,y,z)\,dz$$

$$= \int_{t=a}^{b} A_x(x(t),y(t),z(t))\frac{dx}{dt}\,dt + A_y(x(t),y(t),z(t))\,\frac{dy}{dt}\,dt$$

c) Integrate 1-d integrals using calculus of one variable



* example: line & surface integrals on a paraboloid (Stoke's theorem)

$\vec{A} = yz\hat{x}$ $\quad 0<z<1$

$S:\ z = \frac{1}{4}x^2 + y^2 = s^2(c_\phi^2 + s_\phi^2)$

$\partial S:\ 1 = \frac{1}{4}x^2 + y^2$

$$\begin{aligned} x &= 2s c_\phi & dx &= 2\,ds\, c_\phi + -2s\, s_\phi\, d\phi \\ y &= s\cdot s_\phi & dy &= ds\, s_\phi + s\, c_\phi\, d\phi \\ z &= s^2 & dz &= 2s\cdot ds \end{aligned}$$

$$d\vec{l} = \underbrace{\frac{\partial\vec{r}}{\partial s}}\,ds + \underbrace{\frac{\partial\vec{r}}{\partial\phi}}\,d\phi = d\vec{l}_s + d\vec{l}_\phi$$

$$d\vec{a} = d\vec{l}_s \times d\vec{l}_\phi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2c_\phi & s_\phi & 2s \\ -2s s_\phi & s c_\phi & 0 \end{vmatrix} ds\,d\phi$$

$$= \left(-\hat{x}\, 2s^2 c_\phi - \hat{y}\, 4s^2 s_\phi + \hat{z}\, 2s\right) ds\,d\phi$$

∂S: $\vec{r}(s,\phi)$ $\quad s=1 \quad ds=0 \quad d\vec{l} = d\vec{l}_\phi\ (s=1)$

$$\oint_{\partial S} \vec{A}\cdot d\vec{l} = \int_{\partial S} yz\,dx = -2\int_0^{2\pi} \overbrace{s_\phi^2}^{1/2}\,d\phi = -2\pi$$

$$\int_S \nabla\times\vec{A}\cdot d\vec{a} = \int_S (\hat{y}\partial_z - \hat{z}\partial_y)yz\cdot d\vec{a} = \int_S y\,da_y - z\,da_z$$

$$= \int_0^1\int_0^{2\pi} (s\cdot s_\phi \cdot -4s^2 s_\phi - s^2\cdot 2s)\,ds\,d\phi$$

$$= \int_0^1 ds \int_0^{2\pi} -4s^3 \underbrace{s_\phi^2}_{1/2} - 2s^3\, d\phi$$

$$= \int_0^1 -4s^3\cdot ds\cdot 2\pi = \frac{-4s^4}{4}\Big|_0^1 \cdot 2\pi = -2\pi$$

* alternate method: substitute for dx, dy, dz (antisymmetric)

$$\int_S y\,dz\,dx - z\,dx\,dy = \int_S s s_\phi\cdot 2s\,ds\cdot(2c_\phi\,ds - 2s s_\phi\,d\phi) - s^2(2c_\phi\,ds - 2s s_\phi\,d\phi)(s_\phi\,ds + s c_\phi\,d\phi)$$

$$= \int_S -4s^3 s_\phi^2\,ds\,d\phi - 2s^3 c_\phi^2\,ds\,d\phi + 2s^3 s_\phi^2 \underbrace{d\phi\,ds}_{-ds\,d\phi}$$

$$= \int_S (-6 s_\phi^2 - 2c_\phi^2)\, s^3\,ds\,d\phi$$

Flux, Flow, and Substance

* Differential forms

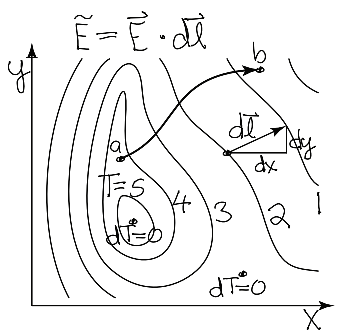
		Name	Geometrical picture
scalar:	$\varphi^{(0)} = \varphi(x)$		level curves
vector:	$\text{đ}\mathcal{E}^{(1)} = \tilde{A} = \vec{A}\cdot d\vec{l} = A_x\,dx + A_y\,dy + A_z\,dz$		equipotentials (flow sheets)
pseudovector:	$\text{đ}\Phi^{(2)} = \tilde{B} = \vec{B}\cdot d\vec{a} = B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy$		fieldlines (flux tubes)
pseudoscalar:	$\text{đ}q^{(3)} = \tilde{\rho} = \rho\,d\tau = \rho\,dx\,dy\,dz$		boxes of substance

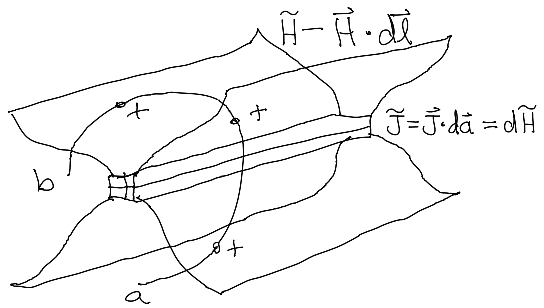
* Derivative 'd'

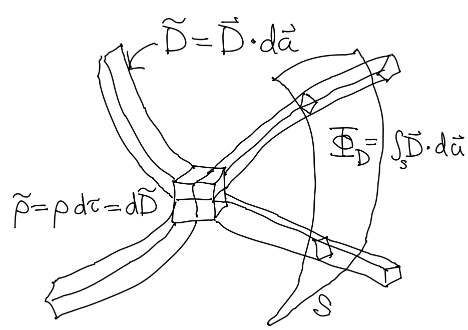
scalar:	$d\varphi = \nabla\varphi\cdot d\vec{l}$	grad	same equipotential surfaces
vector:	$d\tilde{A} = \nabla\times\vec{A}\cdot d\vec{a}$	curl	flux tubes at end of sheets
pseudovector:	$d\tilde{B} = \nabla\cdot\vec{B}\,d\tau$	div	boxes at the end of flux tubes
pseudoscalar:	$d\tilde{\rho} = 0$		

* Definite integral

scalar:			
vector:	$\mathcal{E} = \int_P \tilde{A} = \int_P \vec{A}\cdot d\vec{l}$	flow	# of surfaces pierced by path
pseudovector:	$\Phi = \int_S \tilde{B} = \int_S \vec{B}\cdot d\vec{a}$	flux	# of tubes piercing surface
pseudoscalar:	$Q = \int_V \tilde{\rho} = \int_V \text{đ}q$	subst	# of boxes inside volume







$\Delta f = \int_a^b df = f(b) - f(a) = -4$

$\oint df = \Delta f = 0$

$df = \nabla f\cdot d\vec{l} \quad \vec{E}\cdot d\vec{l} = \tilde{E}$

$\mathcal{E}_H = \int_a^b \tilde{H} = \int_a^b \vec{H}\cdot d\vec{l} = +3$

$\mathcal{E}_H = \oint_{\partial R} \tilde{H} = \int_R d\tilde{H} = \int \tilde{J} = I = +4$

$d\tilde{H} = d(\vec{H}\cdot d\vec{l}) = (\nabla\times\vec{H})\cdot d\vec{a} = \vec{J}\cdot d\vec{a} = \tilde{J}$

$\Phi_D = \int_S \vec{D}\cdot d\vec{a} = \int_S \tilde{D} = +2.$

$\Phi_D = \oint_{\partial R} \tilde{D} = \int_R d\tilde{D} = \int_R \tilde{\rho} = Q = +4$

$d\tilde{D} = d(\vec{D}\cdot d\vec{a}) = \nabla\cdot\vec{D}\,d\tau = \rho\,d\tau = \tilde{\rho}$

* Stoke's theorem

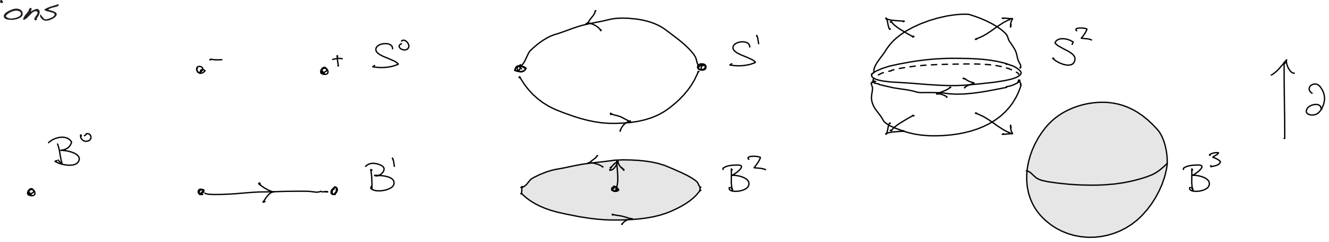
  # of flux tubes puncturing disk (S) bounded by closed path
  EQUALS # of surfaces pierced by closed path (∂S)
  ~ each surface ends at its SOURCE flux tube

* Divergence theorem

  # of substance boxes found in volume (R) bounded by closed surface
  EQUALS # of flux tubes piercin the closed surface (∂R)
  ~ each flux tube ends at its SOURCE substance box

Section 1.3.2-5 – Region | Form = Integral

* Regions



~ definition of boundary operator '∂'
'closed' region (cycle): $\partial S = 0$

~ a boundary is always closed $\boxed{\partial\partial R = 0}$

~ is every closed closed region a boundary?
$$\partial S = 0 \iff S = \partial R$$

~ a room (walls, window, ceiling, floor)
is CLOSED if all doors, windows closed
is OPEN if the door or window is open;

~ what is the boundary?

~ think of a surface that has loops
that do NOT wrap around disks!

* Forms – see last notes

~ combinations of scalar/vector fields and differentials so they can be integrated

~ pictoral representation enables 'integration by eye'

RANK	NOTATION	REGION	VISUAL REP.	DERIVATIVE
scalar	$\omega^{(0)} = f$	Q point	level surfaces	$d\omega^{(0)} = \nabla f \cdot d\vec{l}$
vector	$\omega^{(1)} = \tilde{A} = \vec{A}\cdot d\vec{l}$	P path	flow sheets	$d\omega^{(1)} = \nabla\times\vec{A}\cdot d\vec{a}$
p-vector	$\omega^{(2)} = \tilde{B} = \vec{B}\cdot d\vec{a}$	S surface	flux tubes	$d\omega^{(2)} = \nabla\cdot B \; d\tau$
p-scalar	$\omega^{(3)} = \tilde{\rho} = \rho \, d\tau$	V volume	subst boxes	$d\omega^{(3)} = 0$

edge of the world!

~ properties of differential operator 'd'
transforms form into higher-dimensional form, sitting on the boundary

~ Poincare lemma $\boxed{dd\omega = 0}$ $\qquad \nabla\times\nabla V = 0 \qquad \nabla\cdot\nabla\times\vec{A} = 0$

~ converse – existance of potentials $V, \vec{A}$
$$d\omega = 0 \iff \omega = d\alpha \qquad \nabla\times E = 0 \iff E = -\nabla V \qquad \nabla\cdot\vec{B} = 0 \iff \vec{B} = \nabla\times\vec{A}$$
for space without any n-dim 'holes' in it

* Integrals – the overlap of a region on a form = integral of form over region

~ regions and forms are dual – they combine to form a scalar

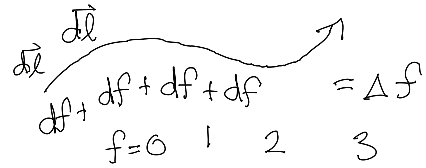
~ generalized Stoke's therem:
'∂' and 'd' are adjoint operators – they have the same effect in the integral

$$\int_R d\omega = \oint_{\partial R} \omega \qquad \text{note: } 0 = \int_{\partial\partial R} \omega = \int_{\partial R} d\omega = \int_R dd\omega = 0$$

Generalized Stokes Theorem

* Fundamental Theorem of Vector Calculus: 0d–1d

$$\int_a^b \nabla\varphi\cdot d\vec{l} = \int_a^b df = f(b)-f(a)$$



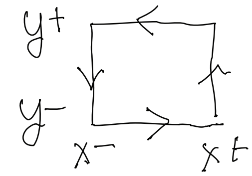
* Stokes' Thereom: 1d–2d

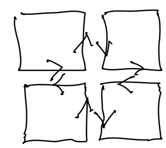
$$\nabla\times\vec{A}\cdot d\vec{a} = \frac{\partial A_y}{\partial x}\,dx\,dy - \frac{\partial A_x}{\partial y}\,dx\,dy + \dots$$

$$= A_y(x^+)\,dy + A_y(x^-)(-dy) + A_x(y^+)(-dx) + A_x(y^-)\,dx + \dots$$

$$= \Sigma\,\vec{A}\cdot d\vec{l} \text{ around boundary}$$

$$+ \text{ other faces}$$



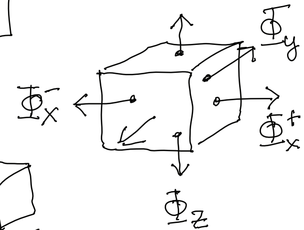


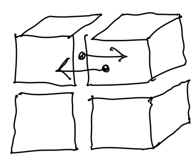
* Gaus' Thereom: 2d–3d (divergence theorem)

$$\nabla\cdot\vec{B}\,d\tau = \frac{\partial B_x}{\partial x}\,dx\,dy\,dz + \frac{\partial B_y}{\partial y}\,dy\,dz\,dx + \frac{\partial B_z}{\partial z}\,dz\,dx\,dy$$

$$= B_x(x^+)\,dy\,dz + B_x(x^-)(-dy\,dz) + 4 \text{ other faces}$$

$$= \Sigma\,\vec{B}\cdot d\vec{a} \text{ around boundary}$$





* note: all interior $f(x)$, flow, and flux cancel at opposite edges
* proof of converse Poincare lemma: integrate form out to boundary
* proof of gen. Stokes theorem: integrate derivative out to the boundary

$$\int_R d\omega = \oint_{\partial R}\omega \iff \int_P \nabla\varphi\cdot d\vec{l} = \oint_{\partial P}\varphi \qquad \int_S \nabla\times\vec{A}\cdot d\vec{a} = \oint_{\partial S}\vec{A}\cdot d\vec{l} \qquad \int_R \nabla\cdot\vec{B}\,d\tau = \oint_{\partial R}\vec{B}\cdot d\vec{a}$$

* example – integration by parts

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}f\right) = \left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f + \frac{\hat{r}}{r^2}\cdot\nabla f$$

$$\int_V \frac{\hat{r}}{r^2}\cdot\nabla f\,d\tau = \int_V \nabla\cdot\left(\frac{\hat{r}}{r^2}f\right)\cdot d\tau - \int_V\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)f\,d\tau$$

$$\int_V \frac{1}{r^2}\frac{\partial f}{\partial r}\,r^2dr\cdot d\Omega = \oint_{\partial V} d\vec{a}\cdot\frac{\hat{r}}{r^2}f - \int_V 4\pi\,\delta^3(\vec{r})\,f\,d\tau$$

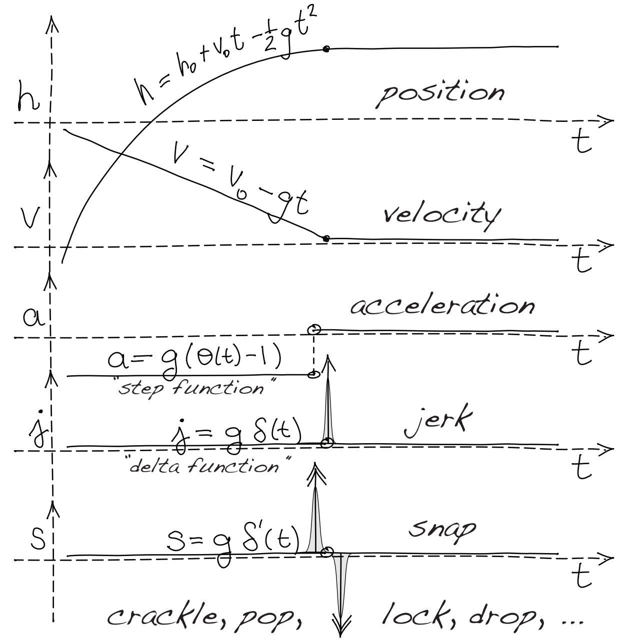
$$\int d\Omega\int_{r=0}^{R} df = \int r^2 d\Omega\,\hat{r}\cdot\frac{\hat{r}}{r^2}f - 4\pi f(0)$$

$$\int d\Omega\, f(R) - f(0) = \int d\Omega\, f(R,\theta,\phi) - 4\pi f(0)$$

$$4\pi\left[\langle f\rangle_R - f(0)\right] = 4\pi\left[\langle f\rangle_R - f(0)\right]$$

Section 1.5 - Dirac Delta Distribution

* Newton's law: yank = mass x jerk
http://wikipedia.org/wiki/position_(vector)



* definition: $d\theta = \delta(x - x')\,dx$ is defined by its integral (a distribution, differential, or functional)

$$\int_a^b \underbrace{\delta(x)\,dx}_{d\theta \text{ "differential"}} = \int_a^b d\theta = \Theta(x)\Big|_a^b = \begin{cases} 1 & a<0<b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$ it is a "distribution," NOT a function!

* important integrals related to $\delta(x)$

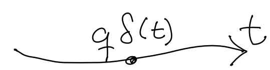
$$\int_{-\infty}^{\infty} \Theta(x)\, f(x)\,dx = \int_0^{\infty} f(x)\,dx \qquad \text{"mask"}$$

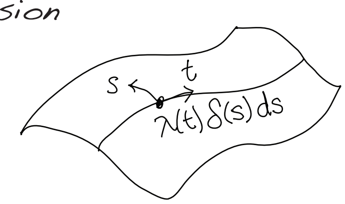
$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\,dx = f(0) \qquad \text{"slit"}$$

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\,dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx = -f'(0)$$

* $\delta(x-x')$ is the an "undistribution" - it integrates to a lower dimension

$$\int_C dq = \int_C \lambda\, dl = \int_C q\, \underbrace{\delta(t)\,dt}_{d\theta} = q$$

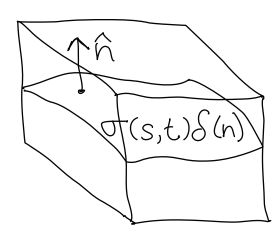




$$\int_A dq = \int_A \sigma\, da = \int_A \lambda(t)\, \underbrace{\delta(s)\,ds}_{d\theta}\, dt = \int_C \lambda(t)\,dt = q$$

$$\int_V dq = \int_V \rho\, d\tau = \int_V \sigma(s,t)\, \underbrace{\delta(n)\,dn}\, ds\, dt = \int_A \sigma\, da = q$$

$$\text{or } = \int_V q\, \delta^3(\vec{r}) = q \qquad \text{or } = \int_V \lambda\, \delta^2(\vec{r}) = q$$



* $\delta(x-x')$ gives rise to boundary conditions - integrate the diff. eq. across the boundary

$$\nabla \cdot \vec{D} = \rho = \sigma(s,t)\,\delta(n)$$

$$\nabla \to \hat{n} \cdot \Delta \qquad \rho \to \sigma \qquad \vec{J} \to \vec{K}$$

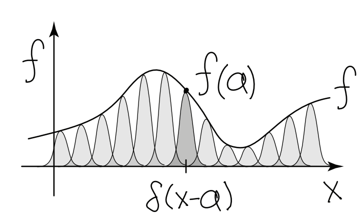
$$\int_{n=0^-}^{0^+} dn \left(\frac{\partial D_n}{\partial n} + \cancel{\frac{\partial D_s}{\partial s}} + \cancel{\frac{\partial D_t}{\partial t}} \right) = \int_{0^-}^{0^+} \sigma(s,t)\,\delta(n)\,dn$$

$$\boxed{\hat{n} \cdot \Delta\vec{D} = \sigma}$$

* $\delta(x-x')$ is the "kernel" of the identity transformation

$$f = I f \qquad f(x) = \underbrace{\int_{-\infty}^{\infty} dx'\, \delta(x - x')}_{\text{identity operator}} f(x')$$

(component form)



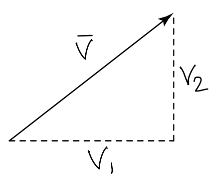
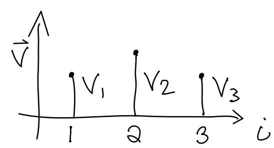
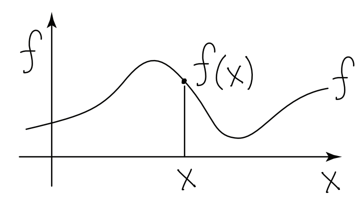
* $\delta(x-x')$ is the continuous version of the "Kroneker delta" δ_{ij}

$$a = I a \qquad a_i = \sum_{j=1}^{n} \delta_{ij}\, a_j \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Linear Function Spaces

* functions as vectors (Hilbert space)

~ functions under pointwise addition have the same linearity property as vectors

	VECTORS	FUNCTIONS
~ addition	$\vec{w} = \vec{v} + \vec{u} \qquad w_i = v_i + u_i$	$h = f + g \qquad h(x) = f(x) + g(x)$
~ expansion	$\vec{v} = \underbrace{\sum_i}_{\text{index}} v_i = \underbrace{v_1}_{\text{component}} \hat{e}_1 + v_2 \underbrace{\hat{e}_2}_{\text{basis vector}} + \ldots$	$f(x) = \int_{x'=-\infty}^{\infty} \underbrace{f(x')}_{\text{component}} \cdot \underbrace{\delta(x-x')}_{\text{basis function}}$ (index: x') or $f(x) = \sum_{i=0}^{\infty} \overbrace{f_i} \cdot \overbrace{\phi_i(x)}$
~ graph	 	
~ inner product (metric, symmetric bilinear product)	$\vec{v} \cdot \vec{u} = \sum_{i=1}^{n} v_i u_i$	$\langle f \mid g \rangle = \int_{-\infty}^{\infty} dx\, f(x)\, g(x)$
~ orthonormality (independence)	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\int_{-\infty}^{\infty} \phi_i(x)\, \phi_j(x) = \delta_{ij} \qquad \int_{x'=-\infty}^{\infty} \delta(x-x')\,\delta(x'-y) = \delta(x-y)$
~ closure (completeness)	$\sum_{i=1}^{n} \hat{e}_i \hat{e}_i \cdot = I$	$\sum_{i=0}^{\infty} \phi_i(x)\, \phi_i(y) = \int_{x'=-\infty}^{\infty} \delta(x-x')\,\delta(x'-y) = \delta(x-y)$
~ linear operator (matrix)	$\vec{u} = A\vec{v} \qquad u_i = A_{ij} v_j$	$f = Hg \qquad f(x) = \int_{-\infty}^{\infty} dx'\, H(x,x')\, g(x')$
~ orthogonal rotation (change of coordinates) (Fourier transform)	$x' = Rx$ $R^T R = I$	$\tilde{f}(k) = \frac{1}{2\pi} \int dx\, e^{ikx} f(x)$ $\int dk\, e^{-ikx} e^{ikx'} = \int dk\, e^{-ik(x-x')} = 2\pi\, \delta(x-x')$
~ eigen-expansion (stretches) (principle axes)	$A\vec{v} = \vec{v}\lambda$ $AV = VW$	$H\phi(x) = \lambda \phi(x)$ (Sturm-Liouville problems)
~ gradient, functional derivative	$\nabla f = \frac{df}{d\vec{r}}$	$\frac{\delta F[\rho(x)]}{\delta \rho}$ (functional minimization)

* Sturm-Liouville equation - eigenvalues of function operators (2^{nd} derivative)

$$\mathcal{L}[y] = -\frac{d}{dx}\left[p(x)\frac{d}{dx}y\right] + q(x) = \lambda\, w(x)\, y \qquad \text{BC: } y(a),\ y(b)$$

~ there exists a series of eigenfunctions $y_n(x)$ with eigenvalues λ_n

~ eigenfunctions belonging to distinct eigenvalues are orthogonal $\langle y_i | y_j \rangle = \delta_{ij}$

Green Functions $G(x, x')$

* Green's functions are used to "invert" a differential operator
~ they solve a differential equation by turning it into an integral equation

* You already saw them last year! (in Phy 232)
~ the electric potential of a point charge

$$\S 1.5.1: \quad \nabla \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

a) $\frac{1}{r^2} \to \infty$ at $r = 0$ "singularity"

b) $$\int_V \nabla \cdot \frac{\hat{r}}{r^2} d\tau = \oint_{\partial V} d\vec{a} \cdot \frac{\hat{r}}{r^2} = \oint_{\Omega} d\Omega\, r^2 \frac{1}{r^2} = 4\pi$$

independent of volume if 0 inside

thus $$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(\vec{r})$$

c) $$\nabla \frac{1}{r} = \hat{r}\frac{\partial}{\partial r}\frac{1}{r} = -\frac{\hat{r}}{r^2}$$

$$\boxed{\begin{array}{ccccc} V & \xrightarrow{-\nabla} & E & \xrightarrow{\nabla \cdot} & \rho/\varepsilon_0 \\ \frac{1}{r} & \to & \frac{\hat{r}}{r^2} & \to & 4\pi \delta^3(\vec{r}) \end{array}}$$

$$-\nabla^2 V = \rho/\varepsilon_0$$
(Poisson equation)

* Green's functions are the simplest solutions of the Poisson equation

$$G(\vec{r}, \vec{r}') \equiv G(\mathscr{r}) = \frac{-1}{4\pi \mathscr{r}} = \nabla^{-2} \delta^3(\vec{\mathscr{r}})$$

~ is a special function which can be used to solve Poisson equation symbolically
using the "identity" nature of $\delta^3(\vec{r} - \vec{r}') = \delta^3(\vec{\mathscr{r}})$

~ intuitively, it is just the "potential of a point source"

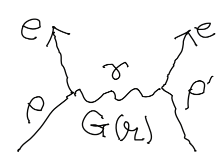
$$\nabla^2 G(\mathscr{r}) = \nabla \cdot \nabla \frac{-1}{4\pi \mathscr{r}} = \nabla \cdot \frac{\hat{\mathscr{r}}}{4\pi \mathscr{r}^2} = \delta^3(\vec{\mathscr{r}}) \qquad \vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$$

let $V = \int_V -G(\mathscr{r}) \frac{\rho(\vec{r}')}{\varepsilon_0} d\tau'$ (solution to Poisson's eq.)

$$\nabla^2 V = \int_{V'} -\frac{\rho(\vec{r}')}{\varepsilon_0} \nabla^2 G(\vec{r} - \vec{r}') d\tau' = \int_{V'} -\frac{\rho(\vec{r}')}{\varepsilon_0} \delta^3(\vec{r} - \vec{r}')\, d\tau' = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

* this generalizes to one of the most powerful methods of solving problems in E&M
~ in QED, Green's functions represent a photon 'propagator'
~ the photon mediates the force between two charges
~ it 'carries' the potential from charge to the other

$$U = \int \rho V d\tau = \iint \rho G \rho' d\tau d\tau'$$

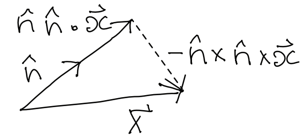


Section 1.6 – Helmholtz Theorem

* orthogonal projections $P_{\parallel}$ and $P_{\perp}$: a vector $\vec{n}$ divides the space X into $X_{\parallel n} \oplus X_{\perp n}$

geometric view: dot product $\hat{n}\cdot\vec{x}$ is length of $\vec{x}$ along $\hat{n}$

Projection operator: $P_{\parallel} \equiv \hat{n}\hat{n}\cdot$ acts on x: $P_{\parallel}\vec{x} = \vec{x}_{\parallel} = \hat{n}\,\hat{n}\cdot\vec{x}$



~ orthogonal projection: $\hat{n}\times$ projects $\perp$ to $\hat{n}$ and rotates by 90°

$$\hat{x}_{\perp} = -\hat{n}\times(\hat{n}\times\hat{x}) = P_{\perp}\vec{x} \qquad P_{\perp} = -\hat{n}\times\hat{n}\times \qquad \boxed{P_{\parallel} + P_{\perp} = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$$

* longtudinal/transverse separation of Laplacian (Hodge decomposition)

$$\boxed{\begin{aligned}\nabla\cdot\vec{F} &= \rho \\ \nabla\times\vec{F} &= \vec{J}\end{aligned}}$$

~ is there a solution to these equations for $\vec{F}(r)$ given fixed source fields $\rho(\vec{r})$ and $\vec{J}(\vec{r})$? YES! (compare HW1 #1)

~ proof: $\nabla^2\vec{F} = \nabla\nabla\cdot\vec{F} - \nabla\times\nabla\times\vec{F}$ (longtudinal/transverse components of ∇)

~ formally, $$\vec{F} = -\nabla\underbrace{\Big(-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho}\Big)}_{V} + \nabla\times\underbrace{\Big(-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\vec{J}}\Big)}_{\vec{A}}$$

$\rho, \vec{J}$ are SOURCES

$V, \vec{A}$ are POTENTIAL

~ what does ∇^{-2} mean? Note that $-\nabla^2\dfrac{1}{4\pi r} = \delta^3(\vec{r})$

~ thus $\nabla^{-2}\delta^3(\vec{r}) = \dfrac{-1}{4\pi r} \equiv G(r)$ (see next page) $\quad G = \dfrac{-1}{4\pi r}$ is Green fn

~ use the δ-identity $\rho(\vec{r}) = \int d\tau'\,\delta^3(\vec{r})\rho(\vec{r}')$

$$V(\vec{r}) \equiv -\nabla^{-2}\rho(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{r})\right)\rho(\vec{r}') = \int d\tau'\,\frac{\rho(\vec{r}')}{4\pi r} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$$

$$\vec{A}(\vec{r}) \equiv -\nabla^{-2}\vec{J}(\vec{r}) = \int d\tau'\left(-\nabla^{-2}\delta^3(\vec{r})\right)\vec{J}(\vec{r}') = \int d\tau'\,\frac{\vec{J}(\vec{r}')}{4\pi r} = \frac{\mu_0}{4\pi}\oint\frac{I\,d\vec{l}}{r}$$

~ thus any field can be decomposed into L/T parts $\boxed{\vec{F} = -\nabla V + \nabla\times\vec{A}}$ with $V, \vec{A}$ defined above

SCALAR POTENTIAL V

* Theorem: the following are equivalent definitions of an "irrotational" field:

a) $\nabla\times\vec{F} = \vec{0}$ curl-less

b) $\vec{F} = -\nabla V$ where $V = \int \dfrac{d\tau'\,\nabla\cdot\vec{F}}{4\pi r}$

c) $V(\vec{r}) = \int_{r_0}^{\vec{r}} -\vec{F}\cdot d\vec{l}$ is independent of path

d) $\oint\vec{F}\cdot d\vec{l} = 0$ for any closed path

* Gauge invariance:

if $\vec{F} = -\nabla V_1$ and also $\vec{F} = -\nabla V_2$

then $\nabla(V_2 - V_1) = 0$ and $V_2 - V_1 = V_0$ is constant

("ground potential")

VECTOR POTENTIAL $\vec{A}$

* Theorem: the following are equivalent definitions of a "solenoidal" field:

a) $\nabla\cdot\vec{F} = 0$ divergence-less

b) $\vec{F} = \nabla\times\vec{A}$ where $\vec{A} = \int\dfrac{d\tau\,\nabla\times\vec{F}}{4\pi r}$

c) $? = \int_S \vec{F}\cdot d\vec{a}$ with ∂S fixed is independent of surface

d) $\oint\vec{F}\cdot d\vec{a} = 0$ for any closed surface

* Gauge invariance:

if $\vec{F} = \nabla\times\vec{A}_1$ and also $\vec{F} = \nabla\times\vec{A}_2$

then $\nabla\times(A_2 - A_1) = 0$ and $A_2 - A_1 = \nabla\lambda(\vec{r})$

("gauge transformation")

Section 2.1 – Coulomb's Law

* Electric charge (duFay, Franklin)
- ~ +,− equal & opposite (QCD: $r+g+b=0$)
- ~ $e = 1.6\times10^{-19}$ C, quantized ($q_n < 2\times10^{-21}\,e$)
- ~ locally conserved (continuity)

> *Seventhly*, Chance has thrown in my Way another Principle, more universal and remarkable than the preceding one, and which casts a new Light on the Subject of Electricity. This Principle is, that there are two distinct Electricities, very different from one another; one of which I call *vitreous Electricity*, and the other *resinous Electricity*. The first is that of Glass, Rock-Crystal, Precious Stones, Hair of Animals, Wool, and many other Bodies: The second is that of Amber, Copal, Gum-Lack, Silk, Thread, Paper, and a vast Number of other Substances.

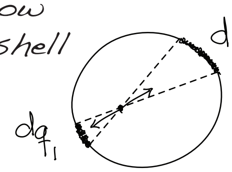
Charles François de Cisternay DuFay, 1734
http://www.sparkmuseum.com/BOOK_DUFAY.HTM

* only for static charge distributions (test charge may move but not sources)

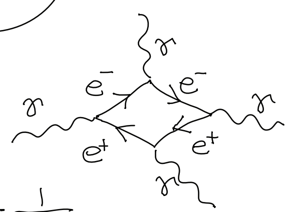
a) Coulomb's law $\vec F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$

b) Superposition $\vec F = \vec F_1 + \vec F_2 + \dots$

- ~ Coulomb: torsion balance
- ~ Cavendish: no electric force inside a hollow conducting shell



- ~ Born-Infeld: vacuum polarization violates superposition at the level of $\alpha^2 = \dfrac{1}{137^2}$



- ~ linear in both q & Q (superposition)
- ~ central force $\vec{\mathscr{r}} \equiv \vec r - \vec r\,'$
- ~ inverse square (Gauss') law $\dfrac{1}{\mathscr{r}^2}$
- ~ units: defined in terms of magnetostatics

$$\varepsilon_0 = 8.85\times10^{-12}\frac{C^2}{Nm^2} = \frac{1}{\mu_0 c^2}$$

$$1\,C \equiv 1\,A\cdot s \qquad F/_{\ell} = 2\times10^{-7}\,N/_{m}$$

(for parallel wires 1 m apart carrying 1 A each)

- ~ rationalized units to cancel 4π in

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta^3(\vec{\mathscr{r}})$$

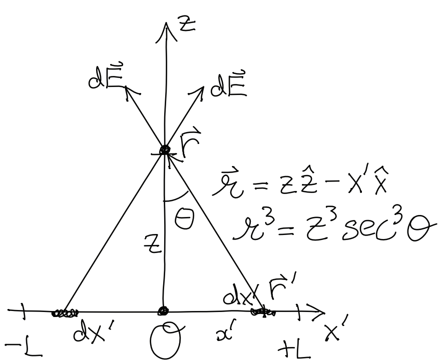
* Electric field
- ~ we want a vector field, but F only at test charge
- ~ action at a distance: the field 'caries' the force from source pt. to field pt.

$$\vec F = \underbrace{\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \dots\right)}Q = Q\vec E$$

$$\vec E = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i\hat{\mathscr{r}}_i}{\mathscr{r}_i^2} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec r\,')\,d\tau'\,\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq'\,\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$dq' \to q_i = q(\vec r_i\,') \text{ or } \lambda(\vec r\,')\,d\ell' \text{ or } \sigma(\vec r\,')\,da' \text{ or } \rho(\vec r\,')\,d\tau'$$

* Example (Griffiths Ex. 2.1)



$$dq' = \lambda\,dx' = \lambda z\sec^2\theta\,d\theta$$

$$\vec E = \frac{1}{4\pi\varepsilon_0}\,2\cdot\int_{x'=0}^{L}\frac{dq'\,\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{1}{4\pi\varepsilon_0}\int_0^L\frac{2\lambda\,dx'\cdot z\hat z}{(z^2+x'^2)^{3/2}} + \overset{(\text{sym})}{0\,\hat x}$$

$$= \hat z\,\frac{2\lambda}{4\pi\varepsilon_0 z}\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$= \hat z\,\frac{2\lambda}{4\pi\varepsilon_0 z}\underbrace{\sin\theta}_{x'/\mathscr{r}}\Big|_{x'=0}^{L}$$

$$= \hat z\,\frac{2\lambda}{4\pi\varepsilon_0 z}\frac{L}{\sqrt{z^2+L^2}}$$

$1+\tan^2\theta = \sec^2\theta$
$x' = z\tan\theta$
$dx' = z\sec^2\theta\,d\theta$
$\mathscr{r}^3 = (z^2+x'^2)^{3/2}$
$= z^3\sec^3\theta$

as $z\to\infty$ $\vec E \approx \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda L}{z^2}$ as $L\to\infty$ $\vec E \approx \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda}{z}$

Section 2.2 - Divergence and Curl of E

* 5 formulations of electrostatics

Coulomb eq. & Superposition

$$\boxed{\vec{E} = \int \frac{dq'\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}} \qquad \vec{F} = q\vec{E} \qquad W = qV$$

Helmholtz

Integral field eq's		Differential field eq's
$\Phi_E = Q/\epsilon_0$	Gauss $\rightleftharpoons$	$\nabla\cdot\vec{E} = \rho/\epsilon_0$
$\mathcal{E}_E = 0$ (closed regions)	Stokes $\rightleftharpoons$	$\nabla\times\vec{E} = \mathbf{0}$
$\updownarrow$		$\updownarrow$ Poincaré
$\mathcal{E}_E = -\Delta V$	FTVC $\rightleftharpoons$	$\vec{E} = -\nabla V$
Potential		Poisson eq.
$V = \int \frac{dq'}{4\pi\epsilon_0 \mathscr{r}}$	Laplace $\rightleftharpoons$ Green	$\nabla^2 V = -\rho/\epsilon_0$

~ all of electrostatics comes out of Coulomb's law & superposition principle

~ we use each of the major theorems of vector calculus to rewrite these into five different formulations
- each formulation useful for solving a different kind of problem

~ geometric pictures comes out of schizophrenetic personalities of fields:

* FLOW (Equipotential surfaces)

$\mathcal{E}_E \equiv \int \vec{E}\cdot d\vec{l}$ ~ integral ALONG the field
~ potential = work / charge

~ $\mathcal{E}_E$ equals # of equipotentials crossed

~ $\Delta\mathcal{E}_E = 0$ along an equipotential surface

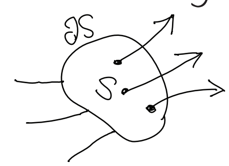
~ density of surfaces = field strength

* FLUX (Field lines)

$\Phi_E \equiv \int \vec{E}\cdot d\vec{l}$ ~ integral ACROSS the field
~ potential = work / charge

$d\Phi = \vec{E}\cdot d\vec{a}$ = # of lines through area

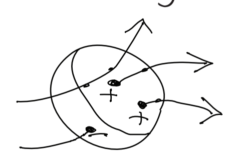
$\vec{E} = \frac{d\Phi}{d\vec{a}}$



~ closed loop

$\int_S d\Phi_E$ = # of lines through loop

~ closed surface

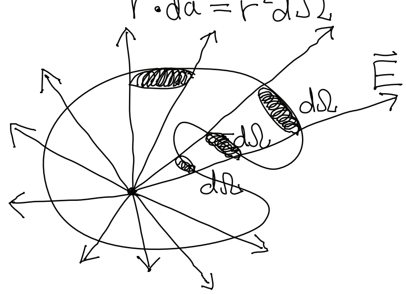


$\int_S d\Phi_E$ = net # of lines out out of surface
= # of charges inside volume

ϵ_0 is unit of proportionality of flux to charge

* Gauss' law

$\hat{r}\cdot d\vec{a} = r^2 d\Omega$



~ solid angle

$d\Omega \equiv \frac{\hat{r}\cdot d\vec{a}}{r^2}$

~ angle (rad.)

$d\vec{\theta} = \frac{\hat{r}\times d\vec{l}}{r}$

~ solid angle of a sphere

$$d\Omega = \sin\theta\, d\theta\, d\phi = -d\cos\theta\, d\phi$$

$$\int\Omega = \int_{\theta=0}^{\pi} -d\cos\theta \cdot \int_{\phi=0}^{2\pi} d\phi = 2\cdot 2\pi = 4\pi$$

~ $\frac{1}{r^2}$ force laws mean there is a const. flux "carrier" field

* Divergence theorem: relationship between differential and integral forms of Gauss' law

$$\Phi_E = \int_{\partial V} \vec{E}\cdot da = \oint \frac{q\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\cdot\hat{\mathscr{r}}\,\mathscr{r}^2 d\Omega = \frac{q}{\epsilon_0} \to \int_V \frac{dq}{\epsilon_0}$$

$$\int_V \nabla\cdot\vec{E}\, d\tau = \int_V \rho/\epsilon_0\, d\tau$$

~ since this is true for any volume, we can remove the integral from each side

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

Section 2.3 – Electric Potential

* two personalities of a vector field: Dr. Jekyl and Mr. Hyde

Flux $= \Phi_E = \int_S \vec{E}\cdot d\vec{a}$ (streamlines) through an area

Flow $= \mathcal{E}_E = \int_P \vec{E}\cdot d\vec{l}$ (equipotentials) downstream

* direct calculation of flow for a point charge

$$\mathcal{E}_E = \int_{\vec{r}=a}^{b} \vec{E}\cdot d\vec{l} = \int_{v'} \frac{dq'}{4\pi\varepsilon_0} \int_{\vec{r}=a}^{b} \frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2}$$

$$= \int_{v'} \frac{dq'}{\varepsilon_0} \left.\frac{1}{4\pi\mathscr{r}}\right|_{\vec{r}=\vec{r}_a}^{\vec{r}_b} \equiv V(\vec{r})\Big|_a^b$$

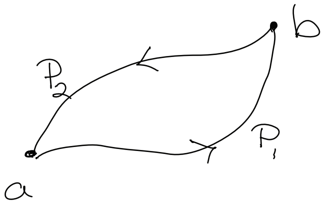
note: this is a perfect differential (gradient)

$$\frac{\hat{\mathscr{r}}\cdot d\vec{l}}{\mathscr{r}^2} = \frac{d\mathscr{r}}{\mathscr{r}^2} = d\frac{-1}{\mathscr{r}}$$

$$df = \nabla f \cdot d\vec{l}$$

$$\nabla \mathscr{r} = \hat{\mathscr{r}}$$

~ open path: note that this integral is independent of path



thus $V(\vec{r}) \equiv -\mathcal{E}_E = \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$ is well-defined

by FTVC: $\Delta V = \int_{\vec{r}_0}^{\vec{r}} \nabla V \cdot d\vec{l}$ so $\boxed{\vec{E} = -\nabla V}$

~ ground potential $V(\vec{r}_0) = 0$ (constant of integration)

~ closed loop (Stokes theorem) $\mathcal{E}_E = \oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a} = 0 \iff \boxed{\nabla\times E = 0}$

for any surface S

* Poincaré lemma: if $\vec{E} = -\nabla V$ then $\nabla\times\vec{E} = -\nabla\times\nabla V = 0$

~ converse: if $\nabla\times\vec{E} = 0$ then $\vec{E} = -\nabla V$ so $\boxed{\vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0}$

* Poisson equation $\nabla\cdot\varepsilon_0 E = \boxed{-\nabla\cdot\varepsilon_0\nabla V = \rho}$ or $\nabla^2 V = \rho/\varepsilon_0$

~ next chapter devoted to solving this equation – often easiest for real-life problems

~ a scalar differential equation with boundary conditions on E_n or V

~ inverse (solution) involves: a) the solution for a point charge (Green's function)

$$V(\vec{r}) = \int_{v'} \frac{dq'}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}} = \int \frac{dq'}{\varepsilon_0} G(\vec{\mathscr{r}}) \quad \text{where} \quad G(\vec{\mathscr{r}}) = \frac{1}{4\pi\mathscr{r}}$$

$$\boxed{\begin{aligned} \nabla^2 G(\vec{\mathscr{r}}) &= \delta^3(\vec{\mathscr{r}}) \\ G(\vec{\mathscr{r}}) &= \nabla^{-2}\delta^3(\vec{\mathscr{r}}) \end{aligned}}$$

$$\nabla^2 G = \nabla\cdot\nabla\frac{1}{4\pi\mathscr{r}} = \nabla\cdot\frac{-\hat{\mathscr{r}}}{4\pi\mathscr{r}^2} = -\delta^3(\vec{\mathscr{r}})$$

b) an arbitrary charge distribution is a sum of point charges (delta functions)

$$\nabla^2 V = \int \frac{dq'}{\varepsilon_0} \nabla^2 G = \int_{v'} \frac{\rho(\vec{r}')\,d\tau'}{\varepsilon_0} \delta^3(\vec{\mathscr{r}}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\boxed{\rho(\vec{r}) = \int_{v'} \rho(\vec{r}')\,d\tau'\,\delta^3(\vec{r}-\vec{r}') = \int_{v'} dq'\,\delta^3(\vec{\mathscr{r}})}$$

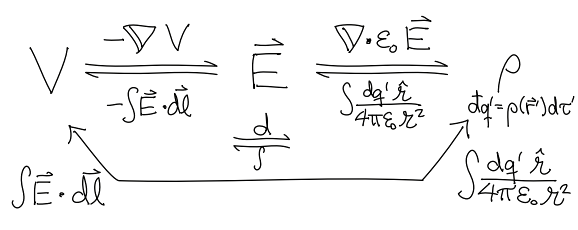
going backwards:

$$V = \nabla^{-2}\frac{\rho(\vec{r})}{\varepsilon_0} = \int_{v'} \frac{\rho(\vec{r}')\,d\tau'}{\varepsilon_0} \nabla^{-2}\delta^3(\vec{\mathscr{r}}) = \int_{v'} \frac{dq'}{\varepsilon_0} G(\vec{\mathscr{r}})$$

~ this is an essential component of the Helmholtz theorem $\boxed{\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times}$

$$\vec{E} = \underbrace{-\nabla(-\nabla^{-2}\overbrace{\nabla\cdot E}^{\rho/\varepsilon_0})}_{V = -\nabla^{-2}\rho/\varepsilon_0 = \int_{v'}\frac{dq'}{4\pi\varepsilon_0\mathscr{r}}} + \nabla\times(-\nabla^{-2}\overbrace{\nabla\times\vec{E}}^{0}) = -\nabla(-\nabla^{-2}\rho/\varepsilon_0) \quad \text{thus} \quad \vec{E} = -\nabla V \iff \nabla\times\vec{E} = 0$$

* derivative chain $\boxed{V \xrightarrow{d} \vec{E} \xrightarrow{d} \rho}$



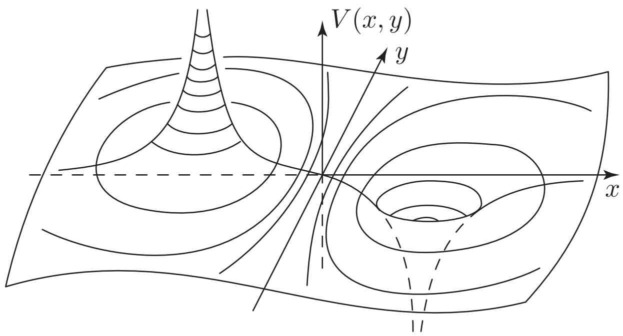
~ inverting Gauss' law is more tortuous path!

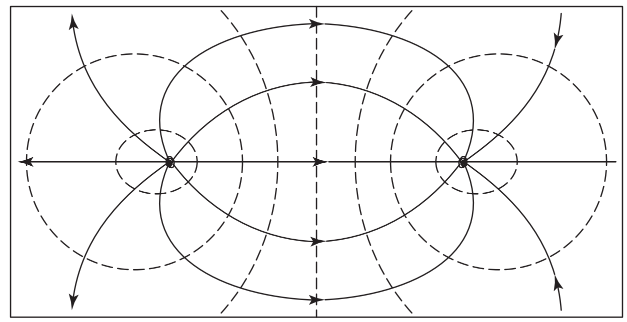
$$\rho \rightarrow V \rightarrow \vec{E} \qquad \vec{E} = -\nabla V = \int \frac{dq'}{4\pi\varepsilon_0} \nabla\frac{-1}{\mathscr{r}}$$

Field Lines and Equipotentials

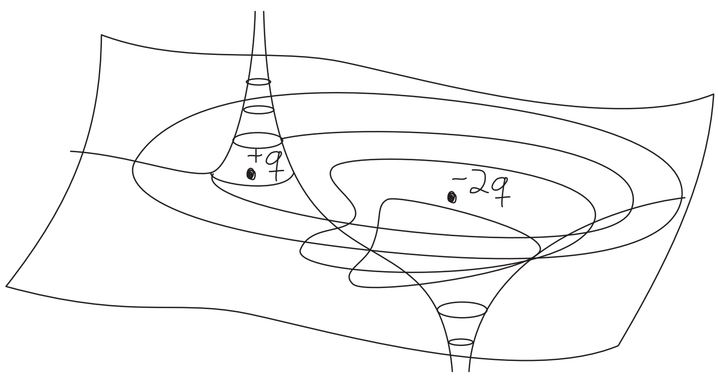
* for along an equipotential surface:
  fo field lines are normal to equipotential surfaces

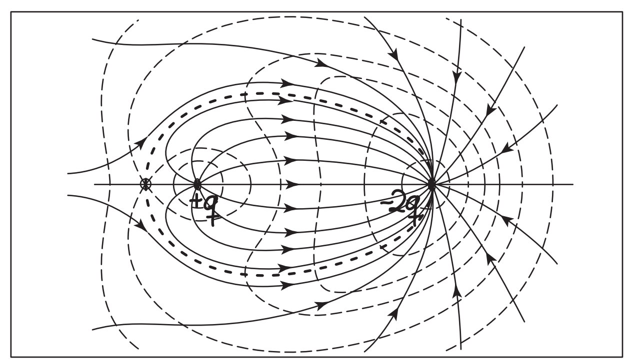
* dipole "two poles" - the word "pole" has two different meanings: (but both are relevant)
  a) opposite (+ vs - , N vs S, bi-polar)
  b) singularity (1/r has a pole at r=0)



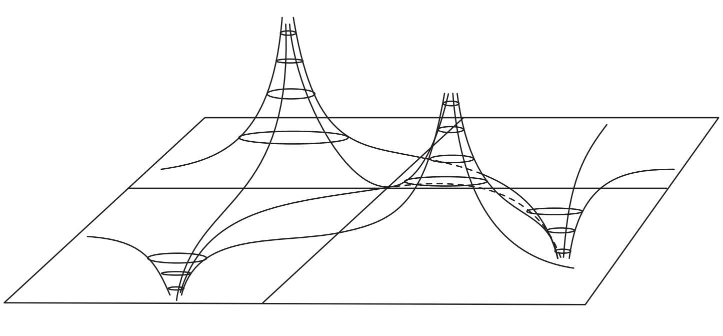


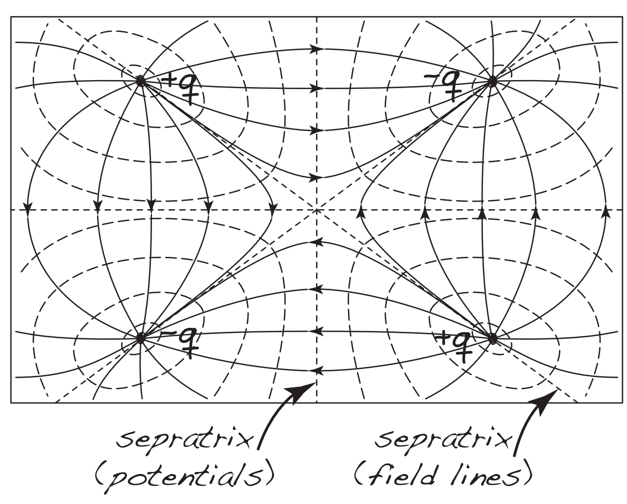
* effective monopole (dominated by -2q far away)





* quadrupole (compare HW3 #2)





Section 2a - Examples

* show that $\nabla\cdot\vec{E}=\rho/\varepsilon_0$ from Coulomb's law

note that $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=\left(\frac{\partial}{\partial(x-x')},\frac{\partial}{\partial(y-y')},\frac{\partial}{\partial(z-z')}\right)=\nabla_{\mathscr{r}}$ (if $\vec{x}'$ fixed)

$$\nabla\cdot\int\frac{dq'\,\hat{\mathscr{r}}}{4\pi\varepsilon_0\mathscr{r}^2}=\nabla\cdot\int_{V'}\frac{\rho(\vec r')\,d\tau'\,\hat{\mathscr r}}{4\pi\varepsilon_0\mathscr r^2}=\frac{1}{4\pi\varepsilon_0}\int_{V'}\rho(\vec r')\,d\tau'\,\nabla_{\mathscr r}\cdot\frac{\hat{\mathscr r}}{\mathscr r^2}$$

$$=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec r')\,d\tau'\,4\pi\delta^3(\vec{\mathscr r})=\rho(\vec r)/\varepsilon_0$$

* derive Coulomb's law from the differential field equations

$\nabla\cdot\vec E=\rho/\varepsilon_0$ $\quad\nabla\times\vec E=0$ $\qquad\nabla^2=\nabla\nabla\cdot-\nabla\times\nabla\times$

$$\vec E=-\nabla\big(\underbrace{-\nabla^{-2}\nabla\cdot\vec E}\big)+\nabla\times\big(-\nabla^{-2}\,\overset{0}{\cancel{\nabla\times\vec E}}\big)=-\nabla\int\frac{d\tau'\,\nabla'\cdot\vec E(\vec r')}{4\pi\mathscr r}=-\nabla\int\frac{d\tau'\rho(\vec r')}{4\pi\varepsilon_0\mathscr r}$$

$$=\int\frac{d\tau'\rho(\vec r')}{4\pi\varepsilon_0}\nabla_{\mathscr r}\frac{-1}{\mathscr r}=\int\frac{d\tau'\rho(\vec r')}{4\pi\varepsilon_0}\frac{\hat{\mathscr r}}{\mathscr r^2}=\int\frac{dq'\,\hat{\mathscr r}}{4\pi\varepsilon_0\mathscr r^2}$$

* show that the differential and integral field equations are equivalent

$\Phi_E=Q/\varepsilon_0\iff\nabla\cdot E=\rho/\varepsilon_0$ $\qquad\Phi_E=\oint_{\partial V}d\vec a\cdot\vec E=\int_V\nabla\cdot\vec E\,d\tau$

~ apply the divergence theorem
~ since Gauss' law holds for any volume,
  it is only true if the integrands are equal

$Q/\varepsilon_0=\int_V\rho/\varepsilon_0\,d\tau$

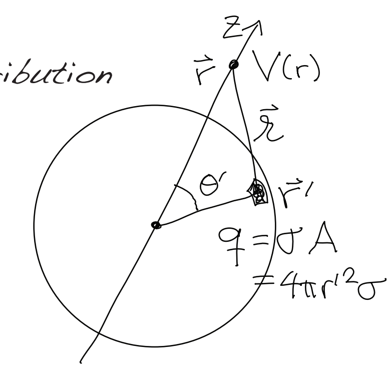
* Griffiths 2.6 find potential of spherical charge distribution

$\int\vec E\cdot d\vec a=\int\rho/\varepsilon_0\,d\tau$ $\qquad 4\pi r^2E(r)=\begin{cases}q/\varepsilon_0 & \text{if } r>r'\\ 0 & \text{if } r<r'\end{cases}$

if $r>r'$ $\quad V(r)=\int_\infty^r-\vec E\cdot d\vec l=\int_\infty^r\frac{-q\hat r}{4\pi\varepsilon_0 r^2}\cdot\hat r\,dr=\frac{q}{4\pi\varepsilon_0}\left.\frac{+1}{r}\right|_\infty^r=\frac{q}{4\pi\varepsilon_0 r}$

if $r<r'$ $\quad V(r)=V(r')+\int_{r'}^r-\vec E\cdot d\vec l=V(r')+\int_{r'}^r0=V(r')$

* Griffiths 2.7 integrate potential due to spherical charge distribution



$4\pi\varepsilon_0V=\int_{sph.}\frac{\sigma\,da'}{\mathscr r}$

$=\int_{u=-1}^{1}2\pi r'^2\sigma\frac{du}{\mathscr r}$

$=\frac{q}{2}\int_{u=-1}^{1}\frac{-d\mathscr r}{rr'}$

$=\frac{q}{2rr'}\left[-|r-r'|+|r+r'|\right]$

$=\frac{q}{2rr'}\begin{cases}-r+r'+r+r' & r>r'\\ +r-r'+r+r' & r<r'\end{cases}$

$da'=r'^2d\Omega'$
$=r'^2\sin\theta'\,d\theta'\,d\phi'$
$=r'^2\cdot -du\,d\phi'$
$u=\cos\theta'$
$-du=\sin\theta'\,d\theta'$

$\mathscr r^2=r^2+r'^2-2rr'u\Rightarrow(r\mp r')^2$
$2\mathscr r\,d\mathscr r=-2rr'\,du$ $\qquad u=\pm1$

$V(r)=\frac{q}{4\pi\varepsilon_0}\cdot\begin{cases}1/r & \text{if } r>r'\\ 1/r' & \text{if } r<r'\end{cases}$

* Griffiths 2.8 find the energy due to a spherical charge distribution

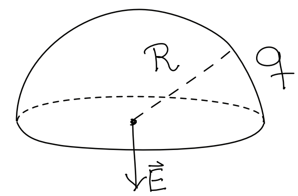
a) $W = \frac{1}{2}\int \sigma \cdot V = \frac{1}{2} q V = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 r'}$

b) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_{r=r'}^{\infty} r^2 dr\, d\Omega \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2$

$= \frac{q^2}{2\cdot 4\pi\varepsilon_0}\int_{r'}^{\infty}\frac{dr}{r^2} = \frac{q^2}{2\cdot 4\pi\varepsilon_0 r'}$

* Quiz: calculate field at origin from a hemispherical charge distribution

$$\vec{E} = \int \frac{dq\, \hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\frac{q}{2\pi} d\Omega\,(-x\hat{x} - y\hat{y} - z\hat{z})}{4\pi\,\varepsilon_0\, R^3} \qquad dq = \frac{q\, d\Omega}{2\pi} = \sigma\, da$$



$$= \frac{-q\hat{z}}{2\pi\cdot 4\pi\varepsilon_0 R^3} \underbrace{\int_{\theta=0}^{\pi/2} R\cos\theta(-d\cos\theta)}_{-R\frac{\cos^2\theta}{2}\Big|_0^{\pi/2} = -\frac{R}{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = \frac{-q\hat{z}}{8\pi\varepsilon_0 R^2}$$

Section 2.4 – Electrostatic Energy

* analogy with gravity

$\vec{F} = q\vec{E}$	$\vec{F} = m\vec{g}$
$W = q\underbrace{Ed}_{\text{potential}=V}$	$W = m\underbrace{gh}_{\text{potential danger}}$

* energy of a point charge in a potential

$$W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\,\Delta V$$

$$W(\vec{r}) = Q\,V(\vec{r}) \qquad V(\infty) \equiv 0$$

* energy of a distribution of charge $q_1, q_2, \ldots$

$$W = \frac{1}{4\pi\varepsilon_0}\left\{ q_2\frac{q_1}{r_{12}} + q_3\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{12}}\right) + q_4\left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right) + \ldots \right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i V_i(\vec{r}_i) \qquad W = \frac{1}{2}\sum q_i V_i$$

* continuous version

$$\sum_{i=1}^{n} q_i \to \int dq$$

$$W = \frac{1}{2\varepsilon_0}\int \rho\,\nabla^{-2}\rho\,d\tau$$

$$W = \frac{1}{2}\int \rho V\,d\tau$$

$$W = \frac{\varepsilon_0}{2}\int E^2\,d\tau$$

* energy density stored in the electric field – integration by parts

$$\nabla\cdot V\vec{E} = \nabla V\cdot\vec{E} + V\nabla\cdot E = -\vec{E}\cdot\vec{E} + V\rho/\varepsilon_0$$

$$0 = \int_{\partial\infty} d\vec{a}\cdot(V\vec{E}) = \int_{\infty}\nabla\cdot V\vec{E} = \int -E^2 + V\rho/\varepsilon_0\,d\tau$$

$$\boxed{\frac{dW}{d\tau} = \frac{\varepsilon_0 E^2}{2}}$$

~ is the energy stored in the field, or in the force between the charges?
~ is the field real, or just a calculational device?
~ if a tree falls in the forest ...

* work does work follow the principle of superposition

~ we know that electric force, electric field, and electric potential do

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = q(\vec{E}_1 + \vec{E}_2 + \ ) = -q\nabla(V_1 + V_2 + \ldots)$$

~ energy is quadratic in the fields, not linear

$$W_{tot} = \frac{\varepsilon_0}{2}\int E^2\,d\tau = \frac{\varepsilon_0}{2}\int E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2\,d\tau$$

$$= W_1 + W_2 + \varepsilon_0\int \vec{E}_1\cdot\vec{E}_2\,d\tau$$

~ the cross term is the 'interaction energy' between two charge distributions
(the work required to bring two systems of charge together)

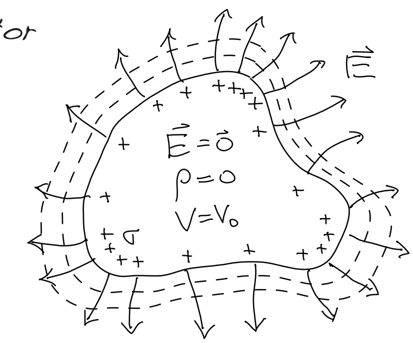
Section 2.5 - Conductors

* conductor
  ~ has abundant "free charge", which can move anywhere in the conductor

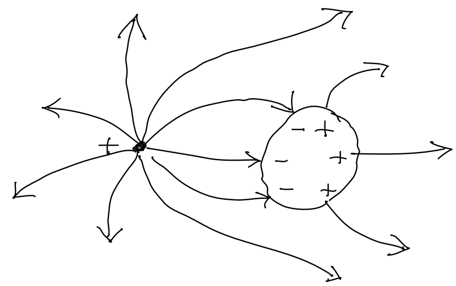
* types of conductors
  i) metal: conduction band electrons, ~ 1 / atom
  ii) electrolyte: positive & negative ions

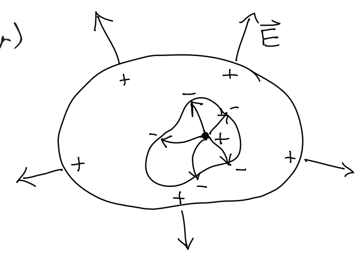
* electrical properties of conductors
  i) electric field = 0 inside conductor
     therefore V = constant inside conductor
  ii) electric charge distributes itself
     all on the boundary of the conductor
  iii) electric field is perpendicular to the
     surface just outside the conductor

	inside	outside
ρ	0	σ
$\vec{E}$	$\vec{0}$	$\frac{\sigma\hat{n}}{\varepsilon_0}$
V	V_0	$V_0+\delta$



* induced charges
  ~ free charge will shift around charge on a conductor
  ~ induces opposite charge on near side of conductor
     to cancel out field lines inside the conductor
  ~ Faraday cage: external field lines are shielded
     inside a hollow conductor
  ~ field lines from charge inside a hollow conductor are
     "communicated" outside the conductor by induction
     (as if the charge were distributed on a solid conductor)
     compare: displacement currents, sec. 7.3





* electrostatic pressure
  ~ on the surface: $\vec{F}/A \equiv \vec{f} = \sigma(\cancel{\vec{E}_{patch}} + \vec{E}_{other}) = \frac{1}{2}\sigma(\vec{E}_{inside} + \vec{E}_{outside})$

  ~ for a conductor: $\vec{E}_{inside} = 0 \quad \vec{E}_{out} = \sigma/\varepsilon_0 \qquad P = f = \frac{\sigma^2}{2\varepsilon_0} = \frac{\varepsilon}{2}E^2$

  ~ note: electrostatic pressure corresponds to energy density
     both are part of the stress-energy tensor $\quad P \approx \omega$

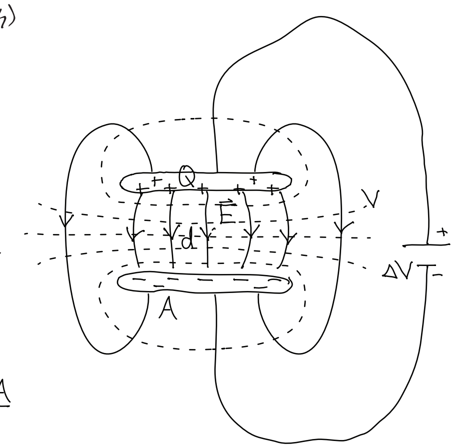
Capacitance

* capacitance
  - ~ a capacitor is a pair of conductors held at different potentials, stores charge
  - ~ electric FLOW from one conductor to the other equals the POTENTIAL difference
  - ~ electric FLUX from one conductor to the other is proportional to the CHARGE

$$C = Q/\Delta V = \frac{\varepsilon_0 \Phi_E}{\mathcal{E}_E} \qquad Q = \int da\, \sigma = \int d\vec{a} \cdot \varepsilon_0 \vec{E} = \varepsilon_0 \Phi_E \quad \text{(closed surface)}$$

$$\Delta V = \int d\vec{l} \cdot \vec{E} = \mathcal{E}_E \quad \text{(open path)}$$

  - ~ this pattern repeats itself for many other components:
    resistors, inductors, reluctance (next sememster)



* work formulation

$$W = \frac{1}{2} QV = \frac{1}{2} CV^2 = \int \frac{\varepsilon_0}{2} E^2 d\tau$$
$$= \frac{\varepsilon_0}{2} \text{flux} \cdot \text{flow}$$
$$C = \frac{2W}{V^2} = \frac{\varepsilon_0}{V^2} \int E^2 d\tau = \frac{\varepsilon_0}{2} \frac{\text{flux} \cdot \text{flow}}{\text{flow} \cdot \text{flow}}$$

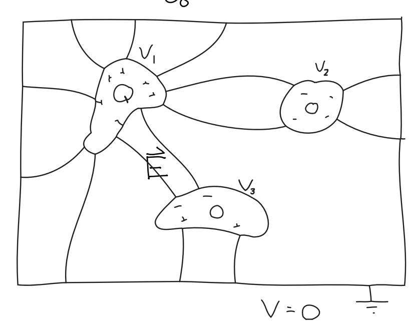
* ex: parallel plates

$$C = \frac{\varepsilon_0 \Phi_E}{\mathcal{E}_E} = \frac{\varepsilon_0 E A}{E d} = \frac{\varepsilon_0 A}{d}$$

* capacitance matrix
  - ~ in a system of conductors, each is at a constant potential
  - ~ the potential of each conductor is proportional
    to the individual charge on each of the conductors
  - ~ proportionality expressed as a matrix
    coefficients of potetial P_{ij} or capacitance matrix C_{ij}

$$V_i = P_{ij}\, Q_j \qquad \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$
$$Q_i = C_{ij}\, V_j$$

$$-\nabla^2 V = \rho/\varepsilon_0 \qquad V(\vec{r}) \propto Q$$



Section 3.1 - Laplace's Equation

* overview: we leared the math (Ch 1) and the physics (Ch2) of electrostatics basically all of the concepts of Phy232, but in a new sophisticated language
  - ~ Ch 3: Boundary Value Problems (BVP) with LaPlace's equation (NEW!)
    a) method of images b) separation of variables c) multipole expansion
  - ~ Ch 4: Dielectric Materials: free and bound charge (more in-depth than 232)

$$\chi \xrightarrow{d} (V,\vec{A}) \xrightarrow{d} (\vec{E},\vec{B}) \xrightarrow{d} 0$$
$$\varepsilon \| \mu \quad \nwarrow \sigma$$
$$(\vec{D},\vec{H}) \xrightarrow{d} (\rho, J) \xrightarrow{d} 0$$

(I) Brute force!
$$\vec{E} = \int \frac{dq'\hat{\mathscr{r}}}{4\pi\varepsilon_0 \mathscr{r}^2}$$

(II) Symmetry
$$\Phi_E = Q/\varepsilon_0$$
$$\mathcal{E}_E = 0$$

(III) Elegant but cumbersome
$$\nabla\cdot\vec{D} = \rho \qquad \text{Ch.4}$$
$$\nabla\times\vec{E} = \vec{0}$$

(IV) Refined brute
$$V = \int \frac{dq}{4\pi\varepsilon_0 \mathscr{r}}$$

(V) the WORKHORSE !!
$$-\nabla^2 V = \rho/\varepsilon \qquad \text{Ch.3}$$

Equations of electrodyamics:

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$	Lorentz force
$\nabla\cdot\vec{J} + \partial_t\rho = 0$	Continuity
$\nabla\cdot\vec{D} = \rho \quad \nabla\times\vec{E} + \partial_t\vec{B} = \vec{0}$	Maxwell electric,
$\nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{H} - \partial_t\vec{D} = \vec{J}$	magnetic fields
$\vec{D} = \varepsilon\vec{E} \quad \vec{B} = \mu\vec{H} \quad \vec{J} = \sigma\vec{E}$	Constitution
$\vec{E} = -\nabla V - \partial_t\vec{A} \quad \vec{B} = \nabla\times\vec{A}$	Potentials
$V \to V + \partial_t\lambda \quad \vec{A} \to \vec{A} + \nabla\lambda$	Gauge transform

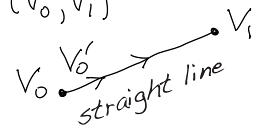
* Classical field equations - many equations, same solution:

Laplace/Poisson:	$\nabla^2 V = 0 \qquad \varepsilon\nabla^2 V = \rho$	~ potentials $(V,\vec{A})$, dielectric ε, permeability μ
Maxwell wave:	$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(V,\vec{A}) - \nabla^2(V,\vec{A}) = \mu(\rho,\vec{J})$	~ speed of light c, charge/current density $(\rho,\vec{J})$
Heat equation:	$C\frac{\partial T}{\partial t} = k\nabla^2 T$	~ temp T, cond. k, heat $\vec{q} = -k\nabla u$, heat cap. C
Diffusion eq:	$\frac{\partial u}{\partial t} = D\nabla^2 u$	~ concentration u, diffusion D, flow $D\nabla u$
Drumhead wave:	$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f$	~ displacement u, speed of sound c, force f
Schrödinger:	$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$	~ prob amp ψ, mass m, potential V, Planck $\hbar$

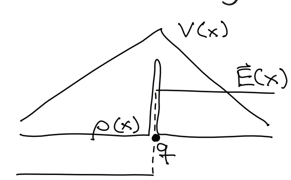
* 1-dimensional Laplace equation $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$

$$\frac{dV}{dx} = \int 0\,dx = a \qquad V = \int a\,dx = ax + b$$

  - ~ a, b satisfy boundary conditions (V_0, V_0') or (V_0, V_1)
  - ~ mean field: $V(x) = \frac{1}{2}(V(x-a) + V(x+a))$
  - ~ no local maxima or minima (stretches tight)

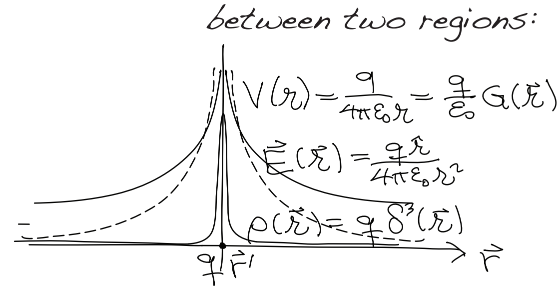


~ charge singularity between two regions:



* 2-dimensional Laplace equation $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$
  - ~ no straighforward solution (method of solution depends on the boundary conditions)
  - ~ Partial Differential Equation (elliptic 2nd order)
  - ~ chicken & egg: can't solve $\frac{\partial^2 V}{\partial x^2}$ until you know $\frac{\partial^2 V}{\partial y^2}$
  - ~ solution of a rubber sheet
  - ~ no local extrema -- mean field: $V(\vec{r}) = \frac{1}{2\pi R}\oint_{\text{circle}} V\,dl$

~ charge singularity between two regions:



* 3-dimensional Laplace equation
  - ~ generalization of 2-d case
  - ~ same mean field theorem: $V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$

Boundary Conditions

* 2nd order PDE's classified in analogy with conic sections: replacing $\frac{\partial}{\partial x}$ with x, etc
  a) Elliptic – "spacelike" boundary everywhere (one condition on each boundary point)
     eg. Laplace's eq, Poisson's eq.
  b) Hyperbolic – "timelike" (2 initial conditions) and "spacelike" parts of the boundary
     eg. Wave equation
  c) Parabolic – 1^{st} order in time (1 initial condition)
     eg. Diffusion equation, Heat equation

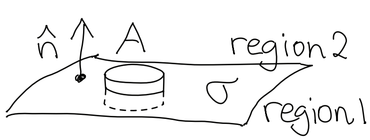
* Uniqueness of a BVP (boundary value problem) with Poisson's equation:
  if V_1 and V_2 are both solutions of $\nabla^2 V = -\rho/\varepsilon_0$ then let $U = V_1 - V_2$ $\quad \nabla^2 U = 0$
  integration by parts: $\nabla\cdot(U\nabla U) = U\,\nabla\cdot\nabla U + \nabla U\cdot\nabla U = U\nabla^2 U + (\nabla U)^2$
  in region of interest: $\oint_{\partial V} d\vec{a}\cdot(U\nabla U) = \int_V \nabla\cdot(U\nabla U)\,d\tau = \int_V U\nabla^2 U + (\nabla U)^2\,d\tau$
  note that: $\nabla^2 U = 0$ and $(\nabla U)^2 > 0$ always
  thus if $\int_{\partial V} d\vec{a}\cdot U\nabla U = \int_{\partial V} da\, \underbrace{U}_{(a)}\, \underbrace{\frac{\partial U}{\partial n}}_{(b)} = 0$ then $\int (\nabla U)^2 d\tau = 0 \Rightarrow U = 0$ everywhere

  a) Dirichlet boundary condition: $U = 0$ – specify potential $V_1 = V_2$ on boundary
  b) Neuman bounary condition: $\frac{\partial U}{\partial n} = 0$ – specify flux $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ on boundary

* Continuity boundary conditions – on the interface between two materials

Flux:

$\vec{D} \equiv \varepsilon\vec{E}$ (shorthand for now)



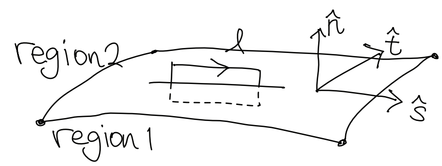
$$\Phi = \oint_{\partial V} \vec{D}\cdot d\vec{a} = \int_V \sigma\, da = Q$$

$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1)\,A = \sigma\cdot A$$

$$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma$$

$$-\frac{\partial V_2}{\partial n} + \frac{\partial V_1}{\partial n} = \sigma/\varepsilon_0$$

Flow:



$\hat{s}\times\hat{t} = \hat{n}$

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = \int_S \nabla\times\vec{E}\cdot d\vec{a}$$

$$\hat{s}\cdot(\vec{E}_2 - \vec{E}_1)\,l = \hat{t}\cdot\nabla\times\vec{E}\;l w = 0$$

$$\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0$$

$$V_2 = V_1$$

* the same results obtained by integrating field equations across the normal

$$\nabla\cdot\vec{D} = \rho/\varepsilon_0$$

$$\int_-^+ dn\left(\frac{\partial D_n}{\partial n} + \frac{\partial D_s}{\partial s} + \frac{\partial D_t}{\partial t}\right) = \int_-^+ dn\,\sigma\,\delta(n)$$

$$\int dD_n = \hat{n}\cdot\Delta\vec{D} = \sigma$$

$$\nabla\times\vec{E} = \vec{K}_e\,\delta(n) \qquad \nabla\times\vec{E} = \begin{vmatrix} \hat{s} & \hat{t} & \hat{n} \\ \partial_s & \partial_t & \partial_n \\ E_s & E_t & E_n \end{vmatrix}$$

$$\int_-^+ dn\left(\hat{t}\,\frac{\partial E_s}{\partial n} - \hat{s}\,\frac{\partial E_t}{\partial n}\right) = \int_-^+ dn\,\vec{K}_e\,\delta(n)$$

$$\hat{n}\times\Delta\vec{E} = \vec{K}_e = 0$$

~ opposite boundary conditions for magnetic fields: $\hat{n}\cdot\Delta\vec{B} = 0 \quad \hat{n}\times\Delta\vec{H} = \vec{K}$