University of Kentucky, Physics 416G
Problem Set #1, due Friday, 2011-09-02

1. a) Show graphically that the following equations define a set of points $\{\boldsymbol{x}\}$ on a line or plane,

	relational	parametric
line	$\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$	$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha$
plane	$\boldsymbol{A} \cdot \boldsymbol{x} = D$	$\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma$

where $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$ are constants, $\boldsymbol{x}_1$ is a fixed point on the line, $\boldsymbol{x}_2$ is a fixed point on the plane, $\boldsymbol{x}$ is an arbitrary point on the line or plane, and α, β, γ are parameters that vary along the line/plane (they uniquely parametrize points in the line/plane). Technically, the set of points $\{\boldsymbol{x} \mid \boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}\}$ or $\{\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{a}\alpha \mid \alpha \in \mathbb{R}\}$ each form a line, while the sets $\{\boldsymbol{x} \mid \boldsymbol{A}\cdot\boldsymbol{x} = D\}$ or $\{\boldsymbol{x} = \boldsymbol{x}_2 + \boldsymbol{b}\beta + \boldsymbol{c}\gamma \mid \beta, \gamma \in \mathbb{R}\}$ each form a plane.

b) What constraints between $\boldsymbol{a}$ and $\boldsymbol{d}$ are implicit in the above formulas, assuming the relational and parametric equations both describe the same line? Likewise, what are the constraints between $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{A}$ if both equations describe the same plane?

c) For the line and the plane, substitute $\boldsymbol{x}$ from the parametric form into the relational form to show that they are consistent. What are $\boldsymbol{d}$ and D in terms of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$?

d) Let us define $\tilde{\boldsymbol{a}} = \boldsymbol{A}/(\boldsymbol{a}\cdot\boldsymbol{A})$. It is parallel to $\boldsymbol{A}$ but "normalized" in the sense that $\boldsymbol{a}\cdot\tilde{\boldsymbol{a}}=1$.

Using the BAC-CAB rule, show that $\boldsymbol{x} = \boldsymbol{a}(\tilde{\boldsymbol{a}} \cdot \boldsymbol{x}) - \tilde{\boldsymbol{a}} \times (\boldsymbol{a} \times \boldsymbol{x})$ for any $\boldsymbol{x}$. This is a projection of $\boldsymbol{x}$ into a vector parallel to the line plus a vector parallel to the plane. Show which term corresponds to each.

e) Using d) calculate the point $\boldsymbol{x}_0$ at the intersection of the line and plane in terms of $\boldsymbol{a}, \boldsymbol{d}, \boldsymbol{A}, D$.

f) Verify e) by showing $\boldsymbol{x}_0$ satisfies the relational equation for both the line and plane.

g) Let $\tilde{\boldsymbol{a}} = \frac{\boldsymbol{b}\times\boldsymbol{c}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$, $\tilde{\boldsymbol{b}} = \frac{\boldsymbol{c}\times\boldsymbol{a}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$, and $\tilde{\boldsymbol{c}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}}$. This definition of $\tilde{\boldsymbol{a}}$ is consistent with above if we require that $\boldsymbol{A} = \boldsymbol{b} \times \boldsymbol{c}$. Calculate the nine combinations of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})^T \cdot (\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$. Note that $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is called the *dual* or *reciprocal* basis of $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ because it is orthonormal to it in the above sense.

h) The *contravariant components* of $\boldsymbol{x}$ are defined as the components (α, β, γ) that satisfy the equation $\boldsymbol{x} = \boldsymbol{a}\alpha + \boldsymbol{b}\beta + \boldsymbol{c}\gamma$. In other words, $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is the contravariant basis. Using $\boldsymbol{x} \cdot \tilde{\boldsymbol{a}}$, etc., calculate the three contravariant components of $\boldsymbol{x}$ in terms of dot products.
Bonus: How is this equivalent to the method of finding components by matrix inversion found in the class notes, first paragraph?

i) Find the *covariant components* $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ of $\boldsymbol{x}$. As above, they are defined as the components $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ which satisfy the equation $\boldsymbol{x} = \tilde{\boldsymbol{a}}\tilde{\alpha} + \tilde{\boldsymbol{b}}\tilde{\beta} + \tilde{\boldsymbol{c}}\tilde{\gamma}$; i.e. $(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{\boldsymbol{c}})$ is the covariant basis.

j) How does everything simplify if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ are an orthonormal basis?

2. The complete vector algebra including dot and cross products can be impleted using the Identity and Pauli matrices for the unit scalar and unit vectors respectively,

$$1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\boldsymbol{x}} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\boldsymbol{y}} = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\boldsymbol{z}} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Both both the dot and cross product are implemented by matrix multiplication. We will prove and work with the formula

$$\sigma_i \sigma_j = I(\sigma_i \cdot \sigma_j) + i(\sigma_i \times \sigma_j) = I\delta_{ij} + i\epsilon_{ijk}\sigma_k. \tag{2}$$

Note the difference between the imaginary i and the index $_i$. Often the I is implied and omitted from formulas.

a) Show that for any product $a \circ b$, the combination $\{a \circ b\} \equiv \frac{1}{2}(a \circ b + b \circ a)$ is symmetric and the combination $\langle a \circ b \rangle \equiv \frac{1}{2}(a \circ b - b \circ a)$ is antisymmetric which respect to exchange of a and b.

b) Show that $\langle a \circ a \rangle = 0$ always. Equivalently, the diagonal elements of an antisymmetric matrix are zero.

c) Show that $a \circ b = \{a \circ b\} + \langle a \circ b \rangle$.

d) Thus we can isolate the dot and cross products as symmetric and antisymmetric portions of the single matrix product. Identify terms of $\sigma_i \sigma_j = \{\sigma_i \sigma_j\} + \langle \sigma_i \sigma_j \rangle$ with Eq. 2.

e) Verify that $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ form an orthonormal basis using the dot product: $\{\sigma_i \sigma_j\} = I\sigma_i \cdot \sigma_j$.

f) Show that $\langle \sigma_i \sigma_j \rangle = i\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j$ for all six combinations of i, j. Thus $\sigma_i \sigma_j$ is antisymmetric for $i \neq j$.

g) The imaginary i in the above formula is not present in the ordinary cross product. This distinguishes pseudovectors from vectors. Calculate the value of the pseudoscalar $\sigma_i \sigma_j \sigma_k$ for all values of i, j, k using the results above to find the distinction between pseudoscalars and scalars.

h) Bonus: These matrices have the additional capability of adding scalars and vectors, which is not meaningful in basic vector spaces. Use this property to simultaneously solve the two equations $\boldsymbol{a} \cdot \boldsymbol{x} = D$ and $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{d}$ from Problem #1 as a single matrix equation, for the case that $\boldsymbol{A} = \boldsymbol{a}$).

3. a) Show that the equation $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{r} = \boldsymbol{0}$ defines a sphere. What is the interpretation of $\boldsymbol{a}$?

b) Show that the 2-d matrix equation $(\boldsymbol{r} - \boldsymbol{r_0})^T A(\boldsymbol{r} - \boldsymbol{r}_0) = d$ defines a conic section (an elipse, circle, parabola, or hyperbola), where $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\boldsymbol{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, and $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. All but $\boldsymbol{r}$ are constant parameters.

c) What is the interpretation of the eigenvectors and eigenvalues of the matrix A?

d) Why doesn't A need an antisymmetric part?

Also, Griffiths chapter 1, problems 1, 4, 5, 8, 10, 12, 15, 16, 18.