University of Kentucky, Physics 416G
Problem Set #2 (Rev. D) due Wednesday, 2011-09-14

1. The **complex plane** is a vector space consisting of the real and imaginary components $\boldsymbol{z} = (x, y)$ of complex numbers $z = x+iy$. In this exercise we will compare complex and vector rotations.

a) Which two complex numbers form the basis of the complex plane in the above components?

b) Show that the dot and cross product of two points $\boldsymbol{z}_1 = (x_1, y_1)$ and $\boldsymbol{z}_2 = (x_2, y_2)$ can be represented by the real and imaginary parts of the complex product $z_1^* z_2 = \boldsymbol{z}_1 \cdot \boldsymbol{z}_2 + i(\boldsymbol{z}_1 \times \boldsymbol{z}_2)_z$, where the $z^* = x - iy$ is called the *complex conjugate* of z. Identify the symmetric and antisymmetric parts of this product. Note 1: this shows that the vector and complex magnitudes coincide, $|z|^2 = z^* z = \boldsymbol{z} \cdot \boldsymbol{z}$. Note 2: for vectors in the xy-plane, only the z-component of the cross product is nontrivial, thus the cross-product is not actually a vector in the original space!

c) Show graphically with coordinates that the operation of multiplying by i rotates a point z by the angle 90° CCW to become the point iz.

d) Show graphically that the operation $z \to (1 + i\,d\theta)z = z + iz\,d\theta$ produces a new point with the same magnitude (assuming $d\theta^{\,2} = 0$), but rotated CCW by the angle $d\theta$.

e) Finite rotations can be obtained by infinite composition of these infinitessimal rotations, which can be solved as follows: Formally integrate the equation $dz = iz\,d\theta$ with the initial condition $z(\theta = 0) = z_0$ to obtain the rotation formula $z(\theta) = R_\theta z_0 = z_0 e^{i\theta}$.

f) Collect the real and imaginary terms of the Taylor expansion of $e^{i\theta}$ to show $e^{i\theta} = \cos\theta + i\sin\theta$.

g) Show that complex multiplication by i is equivalent to the vector operation $\hat{\boldsymbol{z}}\times$.

h) Determine the matrix representation of the operator of $\hat{\boldsymbol{z}}\times$, such that $M_z \boldsymbol{r} = \hat{\boldsymbol{z}} \times \boldsymbol{r}$. Do the same for M_x and M_y. For any vector $\boldsymbol{v}$, show that $\boldsymbol{v}\times = \boldsymbol{M} \cdot \boldsymbol{v} = M_x v_x + M_y v_y + M_z v_z$ and derive the matrix representation of $\boldsymbol{v}\times$. Note that each of these matrices in antisymmetric.

i) Calculate M_x^2, M_y^2, M_z^2, and $M_x M_y$. Note that (M_x, M_y, M_z) are similar to Hamilton's quaternions (i, j, k), which extend the rotational properties of complex numbers to 3 dimensions.

j) Using identifications above, show that matrix for a vector rotation of angle θ CCW in the xy-plane can be written $R_\theta = e^{M_z\theta} = I\cos\theta + M_z\sin\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, neglecting z-components. In general, you have shown that the matrix for a rotation by the angle $v = |\boldsymbol{v}|$ in the CCW direction about the $\hat{\boldsymbol{v}}$-axis can be written as $R_{\boldsymbol{v}} = I\cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}}\sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1 - \cos v)$. The third term deals with the component of the vector that is out-of-plane of the rotation.

k) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\theta} = Ve^{W\theta}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify part j. Thus real symmetric matrices have real eigenvalues while antisymmetric matrices have Tr=0 and imaginary eigenvalues. The exponential of a symmetric matrix remains symmetric, while the exponential of an antisymmetric matrix is an orthogonal rotation with Det=1 and unit length eigenvalues. The symmetric/antisymmetric decomposition of matrices is frighteningly similar to the real/imaginary decomposition of complex numbers.

2. **Affine transformation**. The affine space (points not vectors) is distinguished its lack of significance of the origin. Thus the only two meaningful operations on points are weighed combinations $w_1x_1 + w_2x_2 + \ldots$ of points with total weight $w = w_1 + w_2 + \ldots$ equals to 0 (vector difference of points) or 1 (weighted average). This can be formalized by adding a fourth dimension (weight) which equals 0 for vectors and 1 for points.

a) Show that matrix transformation $T = \begin{pmatrix} R & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{pmatrix}$ translates a point $\begin{pmatrix} \boldsymbol{r} \\ 1 \end{pmatrix}$ by the displacement $\boldsymbol{t}$ and rotation R about the origin. Which operation is done first, in the active sense? What operations are done by the same transformation on vectors $\begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix}$?

b) A robot actuation can be described by the Denavit-Hartenberg parameters (d, θ, r, α), which are described in the video clip http://tekkotsu.no-ip.org/movie/dh-sd.mp4, as a translation d along the z-axis, a rotation θ about the z-axis, a translation r along the new x-axis, and a rotation α about the new x-axis. Build a matrix for each of the above transformations and compose them into a single transformation matrix. Explain how you order the transformations, such that the axes is in the proper location in that sequence of the transformation.

c) Given n measurements $x_i \pm \delta x_i$ of a quantity x with gaussian (rms) error δx, the most accurate estimate of x is made by taking a average of each measurement weighted by $w_i = \delta^{-2} x_i = 1/(\delta x_i)^2$. The uncertainty of the average is calculated from the sum of weights, $w = \sum_i w_i = \delta^{-2} x$. Show how these can be calculated by adding 2-d weighted vectors $w_i(x_i, 1)$.

d) Likewise show how weighted (4-d) vectors $m_i(\boldsymbol{r}_i, 1)$ can be added or integrated to calculate the center of mass of an extended object.

3. **Coordinate basis**

The *coordinate basis* $\boldsymbol{b}_i = \partial \boldsymbol{r}/\partial q^i$, *reciprocal basis* $\boldsymbol{b}^i = \boldsymbol{\nabla} q^i = \partial q^i/\partial \boldsymbol{r}$, and *unit basis* $\hat{e}_i = \boldsymbol{b}_i/h_i$ (where the *scale factor* is $h_i = |\boldsymbol{b}_i|$), are the natural bases to describe components of a vector field within a specific coordinate system (q^1, q^2, q^3). Note that these basis vectors change from one point to the next. Calculate each of the following in both cylindrical and spherical coordinates.

a) Calculate $(\boldsymbol{b}_s, \boldsymbol{b}_\phi, \boldsymbol{b}_z)$ as a functions of $q^i = (s, \phi, z)$ and $(\boldsymbol{b}_r, \boldsymbol{b}_\theta, \boldsymbol{b}_\phi)$ as functions of (r, θ, ϕ).

b) Calculate the resulting scale factors h_i to get unit vectors. Write out the basis transformation matrix in terms of these vectors, i.e. $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})R$.

c) Construct the transformation matrices between unit bases, by considering rotations $R_{\hat{\boldsymbol{z}}}(\phi)$ (rotation about an angle ϕ about the z-axis) and $R_{\hat{\boldsymbol{\phi}}}(\theta)$ and compare with part b.

d) Calculate the reciprocal vectors $(\boldsymbol{b}^s, \boldsymbol{b}^\phi, \boldsymbol{b}^z)$ and $(\boldsymbol{b}^r, \boldsymbol{b}^\theta, \boldsymbol{b}^\phi)$. Show that $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i{}^j$ in general.

e) Calculate the metric $g_{ij} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j$ and $g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j$ in terms of h_i and show that they are inverses.

f) Calculate the line elements $\boldsymbol{dl} = \boldsymbol{b}_i dq^i$, area elements $\boldsymbol{da} = \frac{1}{2} \boldsymbol{dl} \times \boldsymbol{dl}$, and volume elements $d\tau = \frac{1}{6} \boldsymbol{dl} \cdot \boldsymbol{dl} \times \boldsymbol{dl}$. Note that since differentials anticommute, the cross products are not zero.

g) Calculate the derivatives of the basis vectors, $\boldsymbol{\Gamma}_{ij} = \Gamma_{ij}{}^k \boldsymbol{b}_k = \partial \boldsymbol{b}_i/\partial q^j = \partial^2 \boldsymbol{r}/\partial q^i \, \partial q^j$. These *Christoffel symbols* and are needed to calculate derivatives of vectors in curvilinear coordinates.

Also, Griffiths chapter 1, problems 13, 19, 21, 25.