University of Kentucky, Physics 416G
Problem Set #3 (Rev. E) due Friday, 2011-09-23

1. a) Derive formulas for ∇f, $\nabla\times\boldsymbol{A}$, and $\nabla\cdot\boldsymbol{B}$ in an orthogonal curvilinear coordinate system from the definitions $df = \nabla f\cdot\boldsymbol{dl}$, $d(\boldsymbol{A}\cdot\boldsymbol{dl}) = (\nabla\times\boldsymbol{A})\cdot\boldsymbol{da}$, and $d(\boldsymbol{B}\cdot\boldsymbol{da}) = (\nabla\cdot\boldsymbol{B})\cdot d\tau$ (see class notes), using the differential operator $d = dq^i\partial_i$ and differential elements $\boldsymbol{dl} = \hat{\mathbf{e}}_i h_i dq^i$, $\boldsymbol{da} = \frac{1}{2}\boldsymbol{dl}\times\boldsymbol{dl}$, and $d\tau = \frac{1}{3}\boldsymbol{dl}\cdot\boldsymbol{da}$. The results are similar to the cartesian formulas, but with extra scale factors.

b) Expand the formulas from part a) in cylindrical and spherical coordinates to obtain the formulas in the front cover of Griffiths.

c) Show that the trival 2^{nd} derivatives $\nabla\times\nabla f$ and $\nabla\cdot\nabla\times\boldsymbol{A}$ are special cases of the $d^2 = 0$ rule.

d) The Hodge $*$ operator is defined to act on differentials in 3-space by the equations $*1 = dx\,dy\,dz$, $*dx^i = \frac{1}{2}\epsilon_{ijk}dx^j dx^k$, $*dx^i dx^j = \epsilon_{ijk}dx^k$, and $*dx\,dy\,dz = 1$. It converts between differentials for scalars $\leftrightarrow$ pseudoscalars and vectors $\leftrightarrow$ pseudovectors. Calculate $(*dx, *dy, *dz)$, and $(*dy\,dz, *dz\,dx, *dx\,dy)$ to show that $*1 = d\tau$, $*\boldsymbol{dl} = \boldsymbol{da}$, $*\boldsymbol{da} = \boldsymbol{dl}$, and $*d\tau = 1$. What was the equivalent operation in HW1 #2?

e) The *codifferential operator* δ is defined by $\delta \equiv (-1)^p * d\, *$, where p is the dimension of the differential it acts on ($p = 0$ for scalars, $p = 1$ for $\boldsymbol{dl}$, $p = 2$ for $\boldsymbol{da}$, and $p = 3$ for $d\tau$). Show

i) $\delta f = 0$ for a scalar function f,
ii) $\delta(\boldsymbol{A}\cdot\boldsymbol{dl}) = -\nabla\cdot\boldsymbol{A}$ for a vector field $\boldsymbol{A}$,
iii) $\delta(\boldsymbol{A}\cdot\boldsymbol{da}) = (\nabla\times\boldsymbol{A})\cdot\boldsymbol{dl}$, and
iv) $\delta(f\,d\tau) = -(\nabla f)\cdot\boldsymbol{da}$.

Notice that δ is the *adjoint* of d in that it swaps divergence and curl, etc.

f) Show that $\nabla^2 f = -\delta df = -(d\delta+\delta d)f$ for a scalar function f. Note that by mixing differentials and codifferentials, it possible to generate nontrivial second derivatives.

g) Show that the Laplacian of a vector field can be also written as $(\nabla^2\boldsymbol{A})\cdot\boldsymbol{dl} = -(d\delta+\delta d)(\boldsymbol{A}\cdot\boldsymbol{dl})$. Interpret each term in terms of vector derivatives involving ∇.

2. a) Plot a 2-D graph of the function $h(x,y) = 2xy$ by drawing level curves in the xy-plane.

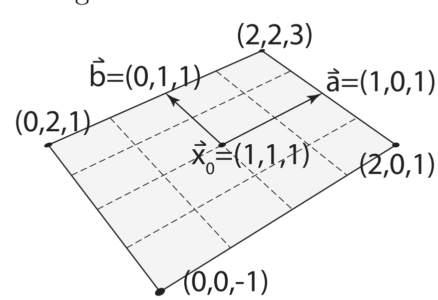
b) Plot a 3-D graph of the 2-D function $h(x,y)$, ie. the surface $\{(x,y,z) \mid z = h(x,y)\}$.

c) Calculate the gradient of h at all points, and the divergence and curl of the gradient.

d) By constructing a 3-D function $g(x,y,z)$ of which $z = h(x,y)$ is a level surface, calculate a normal vector to the 3-D graph $z = h(x,y)$ at all points on the surface. Compare your answer to the normal of the surface calculated with the cross product (calculate $\boldsymbol{da}$ on the surface).

e) Show that there is a linear coordinate transformation $(x,y)\to(x',y')$ that transforms the function $h'(x',y') = y'^2 - x'^2$ into $h(x,y)$. We will find that both of these are $m = 2$ cylindrical multipoles.

f) Show that $h_1(x,y) = 2xy$ and $h_2(x,y) = y^2 - x^2$ form a basis for all functions which can be generated from h_1 and h_2 by rotating the coordinates (x,y) by an angle θ via the transformation $\boldsymbol{r}' = R_\theta\,\boldsymbol{r}$, i.e. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\begin{matrix} c_\phi = \cos\phi \\ s_\phi = \sin\phi \end{matrix}$.



3. a) Integrate $\int_{\mathcal{S}}\boldsymbol{v}\cdot\boldsymbol{da}$ where $\boldsymbol{v} = \hat{\boldsymbol{x}}\,x^2 + \hat{\boldsymbol{y}}\,2yz + \hat{\boldsymbol{z}}\,xy$, and $\mathcal{S}$ is the parallelogram in the figure to the right.

b) Integrate $\oint_{\partial\mathcal{S}}\boldsymbol{v}\cdot\boldsymbol{dl}$ along the boundary of $\mathcal{S}$.

c) Verify Stoke's theorem for the integral in part b.

Also, Griffiths Ch. 1, #28, 29, 31, 32, 33, 34, 35, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 60, 61, 62.