University of Kentucky, Physics 416G
Problem Set #7 (Rev. C), due Friday, 2011-10-28

1. A point charge Q is at the center of a nonconducting spherical shell of radius R with charge $-Q$ uniformly distributed on the surface, and both are placed in an external field $\boldsymbol{E} = E_0\hat{z}$. Solve for the potential and electric field everywhere in space by solving Laplace's equation with the correct boundary conditions for $V_1(\boldsymbol{r})$ inside the sphere and $V_2(\boldsymbol{r})$ outside the sphere.

2. **Sturm-Liouville equation**. We have seen that the functions $\sin(kx)$ and $\cos(kx)$ of Fourier series are orthogonal, and that the Legendre polynomials are orthogonal in spherical coordinates. The purpose of this exercise is to show that this is not an accident, but is true in general for solutions involving separation of variables. Your proof will closely parallel the proof for matrix eigenvalues in the notes; see second page of 2011-08-29 class notes. Given two solutions $L[y_1] = \lambda_1 y_1$ and $L[y_2] = \lambda_2 y_2$ of the differential equation

$$L[y(x)] \equiv \frac{1}{w(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] y(x) = \lambda\, y(x) \tag{1}$$

for $y(x)$ with boundary conditions $y(a) = y(b) = 0$, you will derive the necessary condition for

$$\langle y_1|y_2\rangle \equiv \int_a^b w dx\, y_1 y_2 = 0 \tag{2}$$

Note that L is just a shorthand for the left hand side of the differential equation, but is analogous to a matrix operator, and $\langle y_1|y_2\rangle$ is also a shorthand for the above integral, which is analogous to the vector dot product. y_1, y_2, p, q, and w are all functions of x, and λ_1 and λ_2 are real numbers.

a) Show by integration of parts that $\int_a^b y_1 \frac{d}{dx} y_2 dx = -\int_a^b y_2 \frac{d}{dx} y_1 dx$.

b) Extending part (a), show that $\int_a^b y_1 \frac{d}{dx} p \frac{d}{dx} y_2 dx = \int_a^b y_2 \frac{d}{dx} p \frac{d}{dx} y_1 dx$. Note that the derivative operator acts on everything to the right.

c) Using part (b) show that $\int_a^b w dx\, y_1 L[y_2] - \int_a^b w dx\, y_2 L[y_1] = 0$. We say that the operator L is *self-adjoint* with respect to the weight wdx, analogous to a matrix being symmetric.

d) Evaluate $L[y_1]$ and $L[y_2]$ of the integral in part c) in terms of $\lambda_{1,2}$ using the original differential equation, and factor out the *inner product* $\langle y_1|y_2\rangle$.

e) Conclude that $\langle y_1|y_2\rangle = 0$ if $\lambda_1 \neq \lambda_2$. We say they these *eigenfunctions* are *orthogonal*.

3. **Sagging roof potential** Solve Laplace's equation for the potential $V(x, y)$ defined on the region $-a<x<a$ and $-b<y<b$ with boundary conditions $V(x, \pm b) = 0$ and $V(\pm a, y) = V_0(1-|y/b|)$. Sketch the solution.

4. [bonus] Show that the Rodrigues formula, Griffiths, eq. (3.62) is a solution to the Legendre differential equation (where $x = \cos\theta$)

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_l(x) + l(l+1)P_l(x) = 0. \tag{3}$$

Also Griffiths chapter 3, problems 17, 19, 22, 23, 24.