University of Kentucky, Physics 416G
Problem Set #8 (Rev. B), due Wednesday, 2011-11-09

1. The purpose of this exercise is to develop a more systematic derivation of the multipole potentials than the presenation in the text. At the same time, it is another boundary value problem for more practice. We will solve Laplace's equation for the potential $V(r,\theta)$ inside and outside a spherical shell of radius r'. The surface charge $\sigma(\theta,\phi)$ on the shell is a ring of charge q at the angle $\theta = \theta'$. In the limit $\theta' \to 0$, it is a point charge at the top of the sphere.

a) Write down a formula for $\sigma(\theta)$ of the ring charge using the delta function $\delta(\theta)$. Remember that $\int \sigma\, da = q$ if the region includes the ring charge, and 0 otherwise.

b) Repeat part a) in terms of the variable $x = \cos\theta$. Confirm the the change-of-variables formula for delta functions: $\delta(\theta - \theta')d\theta = \delta(x - x')dx$ where $x' = \cos\theta'$.

c) For the point charge at $\theta = 0$, solve for $V(r,\theta)$ both inside and outside the shell from this charge distribution. Note that $P_\ell(1) = 1$. Compare your answer with $V(r) = q/4\pi\epsilon_0 \mathscr{r}$ to derive the addition formula in Griffiths Eq. 3.94.

d) Repeat for a uniform ring of charge q at the angle θ' on the shell of radius r'.

e) Integrate the potentials by substituting $q \to \int \text{đ}q'$ to obtain the general multipole expansion for an azimuthally symmetric charge distribution as the multipoles $Q^{(\ell)}_{\text{ext,int}}$ discussed in class:

$$V_{\text{ext}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} Q^{(\ell)}_{\text{int}} \frac{P_\ell(\cos\theta)}{r^{\ell+1}} \qquad \text{where} \qquad Q^{(\ell)}_{\text{int}} = \int \text{đ}q' r'^{\ell} P_\ell(\cos\theta') \tag{1}$$

$$V_{\text{int}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} Q^{(\ell)}_{\text{ext}}\, r^{\ell} P_\ell(\cos\theta) \qquad \text{where} \qquad Q^{(\ell)}_{\text{ext}} = \int \text{đ}q' \frac{P_\ell(\cos\theta')}{r'^{\ell+1}} \tag{2}$$

Note that the external multipole potential is only valid outside the entire charge distribution, while the internal multipole potential is only valid completely inside the charge distribution.

f) Identify the terms corresponding to the monopole, dipole, quadrupole, and octupole, terms. What potential is the internal monopole? Match each multipole with the corresponding A_ℓ or B_ℓ coefficient in the general solution from separation of variables.

2. Calculate the leading external and internal multipole (equations 1 and 2) for each charge distribution shown in Fig. 3.27 (the lowest order non-zero multipole). Orient each shape to be centered about the origin with a $+q$ charge on the $+z$-axis, and the opposite vertex on the $-z$-axis. All charges are of magnitude $\pm q$ and the length of each side is a.

Note: even though these distributions do not have perfect azimuthal symmetry, the multipoles $Q^{(\ell)}$ are still well-defined, and can be calculated by the above formulas.

3. a) Calculate the (external) quadrupole tensor for two dipoles $\pm\boldsymbol{p}$ separated by a displacement $\boldsymbol{d}$ from the position of $-\boldsymbol{p}$ to $+\boldsymbol{p}$. Consider two cases: a) $\boldsymbol{p} = p\hat{\boldsymbol{x}}$ and $\boldsymbol{d} = d\hat{\boldsymbol{y}}$; and b) $\boldsymbol{p} = p\hat{\boldsymbol{x}}$ and $\boldsymbol{d} = d\hat{\boldsymbol{x}}$. The other cases would follow the same pattern.

Hint: construct the dipole $\boldsymbol{p}$ with two charges: $+q$ displaced $\boldsymbol{p}/2q$ relative to the center, and $-q$ displaced $-\boldsymbol{p}/2q$ relative to the center. Verify that these two charges have a dipole moment $\boldsymbol{p}$. Put the center of the $+\boldsymbol{p}$ dipole at $\boldsymbol{d}/2$ relative to the origin, and the $-\boldsymbol{p}$ dipole at $-\boldsymbol{d}/2$. Calculate the quadrupole moment of these four point charges.

b) [bonus] calculate the general formula for $\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{d})$ above.

4. [bonus] Make connections between electric quadrupole moments and other concepts:

a) Compare and contrast the quadrupole tensor $\boldsymbol{Q}$ with the moment of inertia tensor $\boldsymbol{\mathcal{I}}$. For example, what is the trace of $\boldsymbol{Q}$ vs. $\boldsymbol{\mathcal{I}}$? How is each used as a tensor?

b) Compare quadrupole potentials with the hydrogen atom 3d orbital. Why are they similar?

5. [bonus] Repeat the expansion in the textbook for the internal dipole at large distances (Example 3.10), to calculate the potential of an external dipole (two point charges $\pm q$ at $\boldsymbol{r} = \pm\hat{\boldsymbol{z}}d/2$), expanded at points (x, y, z) close to the origin, so that $r \ll d$. Show that the result is one of the terms in the solution to $\nabla^2 V = 0$ in spherical coordinates (problem #1). An example of an external dipole is a capacitor, which produces a constant electric field.

Also Griffiths chapter 3, problems 33, 38, 41, 45.