

University of Kentucky, Physics 416G
Problem Set #9 (Rev. A), due Monday, 2011-11-21

1. A spherical capacitor of radius R is made of two hemispherical conductors separated by a small gap. The top hemisphere is held at potential $+V_0$ and the bottom hemisphere at $-V_0$. Evaluate the first two non-zero terms in each case.

a) Calculate the potential everywhere inside the sphere. Note that this solution is independent of the potential outside the sphere, because of the boundary condition $V = \pm V_0$ at $r = R$.

b) Calculate the potential everywhere outside the sphere, which is likewise independent of part a).

c) Calculate the surface charge on both the inside and outside of the sphere.

d) Integrate the total charge on either hemisphere to calculate the capacitance.

e) Draw the equipotentials and field lines, demonstrating the dominant features of the lowest non-zero multipole, both inside and outside the sphere.

f) If the capacitor is filled inside with a dielectric of relative strength ϵ_r , how are the field and surface charge modified inside? On the outside? Calculate the capacitance with the dielectric.

g) [bonus] Sum the terms of all orders to derive the exact formula for the capacitance.

2. The **Clausius-Mossotti** relation. Our purpose is to derive the relation between atomic polarizability α (defined by $\mathbf{p} = \alpha\mathbf{E}$ in a linear dielectric) and electric susceptibility χ_e (defined by $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ in the same medium).

a) The simplest relation can be derived by ignoring the contribution from individual dipoles to \mathbf{E} . Using the definition of \mathbf{P} , explain why $\mathbf{P} = N\mathbf{p}$, where N is the number density of molecules (number/volume). Show that in this case $N\alpha = \epsilon_0\chi_e$.

b) Now correct for the fact a dipole does not feel its own field. Thus the definition of polarizability in a dielectric material must be modified to $\mathbf{p} = \alpha\mathbf{E}_0$, where \mathbf{E}_0 is the macroscopic field due to everything but \mathbf{p} . Calculate \mathbf{E}_0 at the location of \mathbf{p} as a function of the macroscopic field \mathbf{E} and susceptibility χ_e assuming that \mathbf{p} is at the center of a bubble of radius R (the size of the molecule) inside the dielectric material. Hint: see example 4.2 and problem 4.16.

c) Using the modified equation from part a), $N\alpha\mathbf{E}_0 = \epsilon_0\chi_e\mathbf{E}$, and part b), derive the relation $N\alpha = \epsilon_0\chi_e\frac{3}{3+\chi_e} = 3\epsilon_0\frac{\epsilon_r-1}{\epsilon_r+2}$. Show that this reduces to the naïve relation in a) as $\chi_e \rightarrow 0$.

d) Use part a) to compare the α and χ_e for elements which appear in both tables 4.1 and 4.2. Calculate the fractional correction term $\frac{3}{3+\chi_e}$ derived in part c).

3. The **Langevin** formula. In this problem we calculate the electric susceptibility χ_e of a polar material, composed of molecules with permanent dipole moment p . While the dipole has a tendency to align along the total field \mathbf{E} , thermal agitations prevent perfect alignment. Ludwig Boltzman showed that the number of atoms occupying a state i of energy U_i is proportional to the Boltzman factor $w = e^{-U_i/kT}$, where k is the Boltzman constant and T the temperature of the system. Thus colder systems are more likely to settle into the lowest energy state because the probability of being in an excited state is exponentially small. In our case, the state of a dipole in an electric field is given by its rotational direction $i \equiv (\theta, \phi)$ with respect to the field $\mathbf{E} = E\hat{z}$.

- a) Calculate the potential energy $U(\theta)$ of the dipole \mathbf{p} in the field \mathbf{E} .
- b) By considering the solid angle of a ring of constant energy (ie. constant θ), show that the number of states dn of energy between $U(\theta)$ and $U(\theta + d\theta)$ is proportional to $\sin\theta d\theta$. The ratio $g = \frac{dn}{d\theta}$ is called the *degeneracy of states*, the *density of states*, or the *phase space*.
- c) Calculate the weighted average energy $\langle U \rangle \equiv \int dn w(\theta)U(\theta) / \int dn w(\theta)$ of a dipole p in the field E , weighted by the Boltzman factor (distribution).
- d) Calculate the weighted average $\langle p_z \rangle$ of $p_z = p \cos\theta$ as a function of the electric field E and temperature T , also weighted by the Boltzman distribution. Explain why $P = N\langle p_z \rangle$ and use it to show that $P = Np[\coth(pE/kT) - (kT/pE)]$. Graph this as a function of E .
- e) In the limit where $pE \ll kT$, determine the linear relationship between E and P to determine the susceptibility χ_e . Calculate χ_e for liquid and vapor water, using $p = 3.8 \times 10^{-9}$ e cm.
- f) [bonus] As in #2, correct for the fact a dipole does not feel it's own field. How much does χ_e improve for liquid water?

Also Griffiths chapter 4, problems 1, 4, 5, 6, 7, 8, 9, 10, 12, 13, and 14.