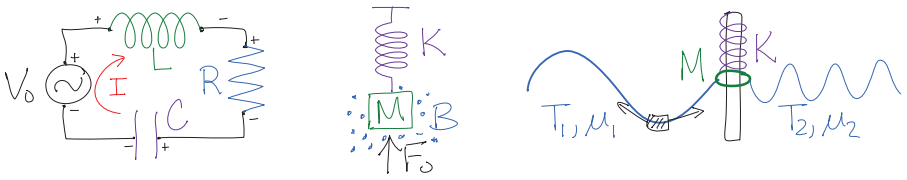
University of Kentucky, Physics 417G
Problem Set #8, due Monday, 2015-03-30

1. **Mechanical impedance** $Z = F/v$ is the ratio of force to velocity of an oscillator, analogous to electrical impedance $Z = V/I$ or E/H. The power $P = Fv$ transmitted by the force involves the same quantities, just like the electrical power $P = VI$ or the Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. See also acoustic, seismic, surface, gravity, quantum, and gravitational impedance.

a) For a mass M on a spring with stiffness K, damping coefficient B, and driving force $\tilde{F}_0 e^{iwt}$ with complex amplitude $\tilde{F}_0$, Newton's law is $\tilde{F}_0 = K\tilde{x} + B\tilde{v} + M\tilde{a}$. Compare the complex mechanical impedance of this oscillator with that of an LRC-circuit. For an undriven, undamped circuit, show that the resonant frequency is $\omega_0 = \sqrt{K/M}$, and at that frequency, the impedance of both the mass and spring are $Z_0 = \sqrt{KM}$. This *characteristic impedance* Z_0 corresponds to the oscillating power $P = Z_0|\tilde{v}|^2 \cos^2(\omega t)$, transmitted between kinetic and potential energy of the system.

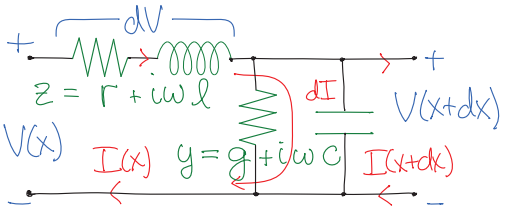


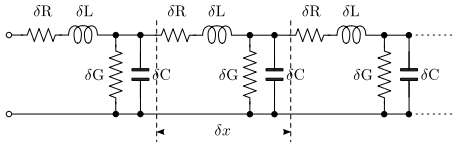
b) In continuous media, the wave equations involve (F, v), (V, I), or (E, H). These fields factor out of the boundary conditions, except for the ratio $Z = F/v$. Thus this *characteristic impedance* of the medium affects reflection and transmission of power. For a string of tension T and mass density μ, show that the transverse force between adjacent segments of the string, which transfers energy and momentum along the line, is $F = Tf'$. Derive the wave equation from Newton's law $dF = \mu dx\, \ddot{f}$ to show that the phase velocity is $v = \sqrt{T/\mu}$. For a sinusoidal wave $f(x,t) = \tilde{A}e^{ikx - i\omega t}$, show that the impedance is $Z \equiv F/\dot{f} = \sqrt{T\mu}$. Show that the kinetic energy density is $u_\mu = \frac{1}{2}\mu \dot{f}^2$ and the potential energy density is $u_T = \frac{1}{2}Tf'^2$. Show that the two are equal so that $u = u_\mu + u_T = Tf'^2 = \mu \dot{f}^2$, and that the power transmitted along the string is $S = F\dot{f} = uv = Z\dot{f}^2$.

c) Two strings of tension $T_{1,2}$ and density $\mu_{1,2}$ are joined by a spring-loaded ring sliding on a rod as in the figure, so that each string can be at a different tension. Show that the two boundary conditions are $\Delta f = 0$ and $\Delta F = \Delta(Z\dot{f}) = Z_\partial \dot{f}$, where Z_∂ is the mechanical impedance of the ring. Solve for the power reflection and transmission coefficients at the ring interface.

d) We can also apply the concept of impedance to elastic collisions, interpreting the boundary conditions as conservation of momentum $p = mv$ and kinetic energy $E = \frac{1}{2}pv$. Here the impedance is $m = p/v$. For two particles of mass m_1 and m_2, initial velocities v_1 and v_2, and final velocities v_1' and v_2', factorize the energy equation $\frac{1}{2}m_1(v_1^2 - v_1'^2) = \frac{1}{2}m_2(v_2'^2 - v_2^2)$ and use conservation of momentum to show that $v_1 + v_1' = v_2 + v_2'$, and therefore $\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_1-m_2 & 2m_2 \\ 2m_1 & m_2-m_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. In particular, if two particles' impedances match, their velocities are exchanged during the collision [see Newton's cradle]. For $v_2 = 0$, show the that the *reflection coefficient* is $R \equiv E_2/E_0 = (m_1 - m_2)^2/(m_1 + m_2)^2$ and the *transmission coefficient* is $T \equiv E_2/E_0 = 4m_1m_2/(m_1 + m_2)^2$.

2. The **Telegrapher's Equations** describe the propagation of electrical signals along a transmission line. In the figure below, the transmission line is modeled by a network of oscillator circuits of length dx with *series impedance* $z = r + i\omega\ell$ and *shunt admittance* $y = g + i\omega c$, where the *primary line constants* are loop resistance r and inductance ℓ, and insulator conductance g and capacitance c per unit length. The voltage between the two lines at any point is $V(x,t) = \tilde{V}e^{\gamma x + i\omega t}$ and the current flowing forward/backward along the top/bottom line is $I(x,t) = \tilde{I}e^{\gamma x + i\omega t}$, where $\gamma = \alpha + i\beta$ is the *propagation constant* ($\alpha = -\kappa =$ attenuation, $\beta = -k =$ spatial frequency).





a) Show from Kirchhoff's laws that $dV = -zI\,dx$ and $dI = -yV\,dx$. Take the ratio of these two equations to derive the *characteristic impedance* of the line Z. Show that the propagation constant is $\gamma = \sqrt{zy}$. Show that for a lossless transmission line ($r = g = 0$), the impedance is $Z = \sqrt{\ell/c}$ and the transmission velocity is $v = \omega/k = 1/\sqrt{\ell c}$, independent of frequency. Otherwise, show that Z is real and independent of frequency if the Heaviside condition, $\ell/r = c/g$ is satisfied, which minimizes dispersion along the line.

b) Show that the power transmitted along the line is $P = I^*V = Z|I|^2$. Calculate the reflection and transmission coefficients at the connection between two transmission lines of impedance Z_1 and Z_2, respectively.

c) Extend the boundary conditions to calculate the transmission and reflection coefficients if we place a shunt resistor of resistance R_0 between the two lines at the connection. With what value of R_0 should one terminate a cable to prevent reflections?

Also, Griffiths 3ed[4ed] Ch. 9, #2[2], 3[3], 4[4], 5[5], 9[9], 11[12].