L02-Planck's law again

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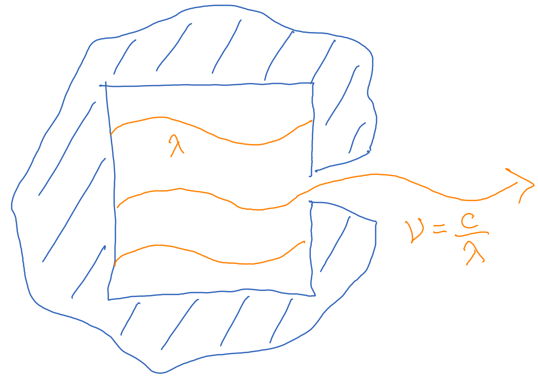
Rayleigh / Jeans were able to explain the low-wavelength tail of the black body spectrum using basic physical principles:

Let, for simplicity, the black body be a cubic cavity with a small hole for the radiation to escape.



They assumed 1) the radiation was present in "modes" inside the cavity which satisfied boundary conditions.

2) the energy of each mode was random, and thermally distributed

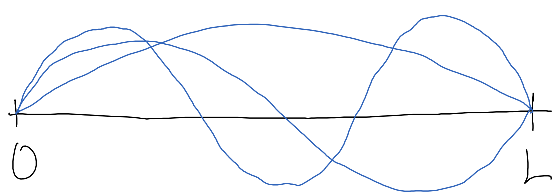
The spectral energy density, defined as $u(\nu) = \dfrac{\text{energy}}{\text{volume} \cdot \Delta\text{frequency}}$ is given by the product:

$$u(\nu) = \underbrace{g(\nu)}_{\#\text{ of modes between } \nu \text{ and } \nu + d\nu} \cdot \underbrace{\bar{E}(\nu)}_{\text{average energy of each mode}}$$

<u>Assumption #1</u>: modes in the cavity:

This is a property of classical waves, but we will apply the same idea to quantum wave functions next class.

1) Consider a 1-d string: clamped at $x=0$ & $x=L$



There are n half-wavelengths

$L = n\dfrac{\lambda}{2}$ or since $c = \lambda\nu$ $\quad \nu = \dfrac{c}{\lambda} = \dfrac{cn}{2L}$

2) Now consider a 2-d square drumhead. The modes will oscillate between evenly spaced node lines:

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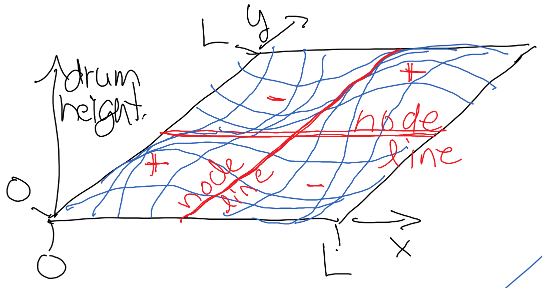
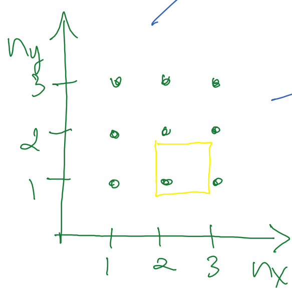
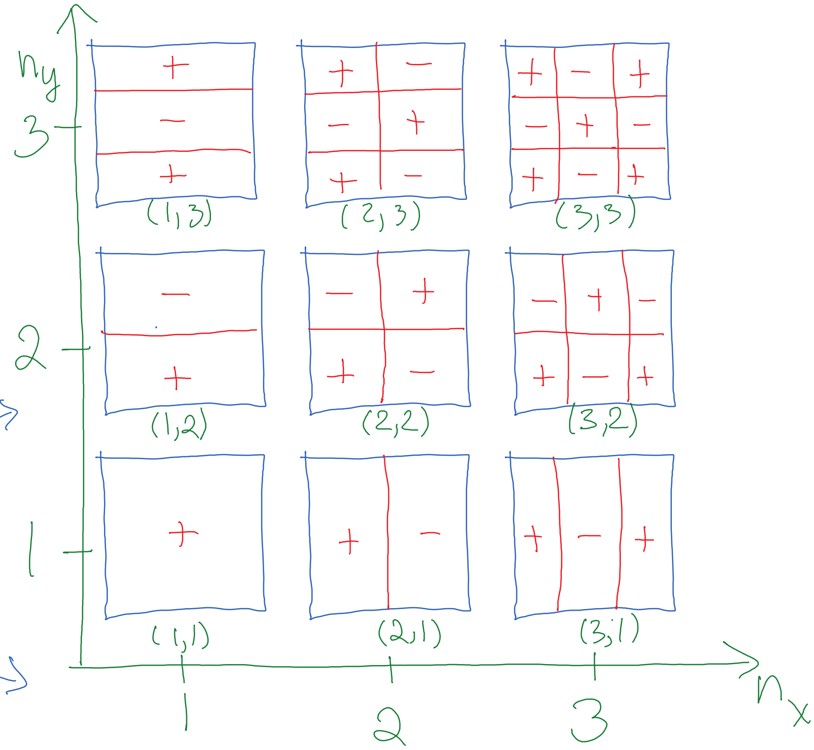


(2,2)-mode

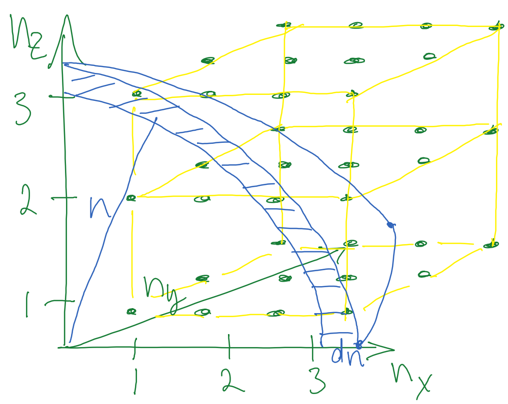
Each dot represents one mode of vibration.





Note each dot occupies one unit of area in the (n_x, n_y) space.

3) In our 3-d cavity, modes have three sets of node planes. They are labelled (n_x, n_y, n_z) and have frequency $\nu = \frac{cn}{2L}$ where $n^2 = n_x^2 + n_y^2 + n_z^2$



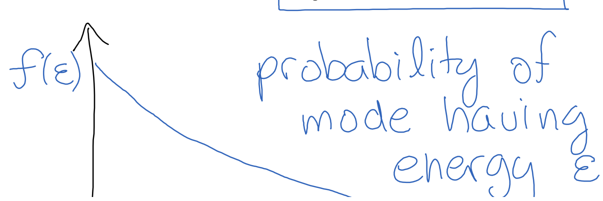
The # of modes between ν & $\nu + d\nu$ is the volume of a shell of radius n and thickness dn

$$\left[g(\nu)\,d\nu = \frac{1}{8} 4\pi n^2 \cdot dn = \frac{\pi}{2}\left(\frac{2L}{c}\right)^3 \nu^2 d\nu \right] \times 2 \text{ polarizations of light}$$

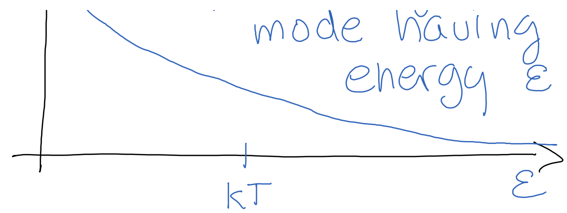
Note the number of modes is much larger at high ν.

Assumption #2: The thermal energy of each mode follows the Boltzmann distribution. $\boxed{f_B(\varepsilon) = e^{-\varepsilon/kT}}$

Lower energy states are exponentially more likely. The distribution broadens



exponentially more likely.
The distribution broadens at higher temperature to accommodate more energy



To calculate $\bar{\varepsilon}(\nu)$ we must take a weighted average: $\bar{\varepsilon}(\nu) = \frac{\int \varepsilon\, f_B(\varepsilon) d\varepsilon}{\int f_B(\varepsilon) d\varepsilon}$

The denominator (normalization) is called the "Partition function"

$$Z(\beta) = \int_0^\infty e^{-\beta\varepsilon} d\varepsilon = \frac{-1}{\beta} e^{-\beta\varepsilon}\Big|_0^\infty = \frac{1}{\beta} \qquad \text{where } \beta \equiv \frac{1}{kT}$$

There is trick to get the numerator from Z:

$$\int \varepsilon\, e^{-\varepsilon/kT} d\varepsilon = \int -\frac{\partial}{\partial\beta} e^{-\beta\varepsilon} = -\frac{\partial}{\partial\beta} Z(\beta) = \frac{1}{\beta^2}$$

Thus $\bar{\varepsilon}(\nu) = -\frac{\partial}{\partial\beta} \ln Z = \frac{-\frac{\partial Z}{\partial\beta}}{Z(\beta)} = \frac{1/\beta^2}{1/\beta} = \frac{1}{\beta} = kT$ independent of ν

This is the equipartion, which states the average energy per degree of freedom (D.O.F.) is $\frac{1}{2}kT$.
The two D.O.F.'s are electric & magnetic fields.

Thus $u_\nu(\nu) = \frac{8\pi\nu^2}{c^3} kT = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{x}$ where $x = \frac{h\nu}{kT} = \frac{hc}{\lambda kT}$

How can we restore the missing UV cutoff at high frequency, and prevent infinite energy?

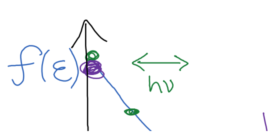
Planck was able to explain his fit to the data

$u_\nu(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^x - 1}$ by making one additional assumption

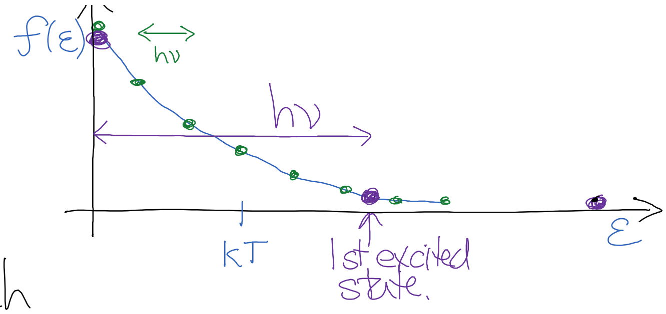
Assumption #3: energy is limited to discrete packets or "quanta" which get larger with frequency $\boxed{\varepsilon = h\nu}$

Instead of integrating $\int d\varepsilon$, he summed over steps $\varepsilon_n = nh\nu$

For low frequency packets,
$\varepsilon = h\nu$ the sum is close



For low frequency packets, $\varepsilon = h\nu$, the sum is close to the previous integral.



For high frequency packets $\varepsilon = h\nu$, there is not enough thermal energy to excite even the first excited state, and thus the average energy of the mode is 0.

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\varepsilon_n/kT} = \sum_{n=0}^{\infty} e^{-\frac{nh\nu}{kT}} = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad a = e^{-\beta h\nu} = e^{-x}$$

$$\sum_{n=0}^{\infty} \varepsilon_n e^{-\varepsilon_n/kT} = -\frac{\partial}{\partial\beta} Z(\beta) = \frac{-1}{(1-a)^2} \cdot e^{-\beta h\nu} \cdot (-)h\nu$$

$$\text{so } \bar{\varepsilon}(\nu) = \frac{\sum_{n=0}^{\infty} \varepsilon_n f(\varepsilon_n)}{\sum_{n=0}^{\infty} f(\varepsilon_n)} = \frac{-\frac{\partial}{\partial\beta} Z(\beta)}{Z(\beta)} = \frac{h\nu e^{-x}}{1-e^{-x}} = \frac{h\nu}{e^x - 1}$$

and $u(\nu) = \dfrac{8\pi h\nu}{c^3} \cdot \dfrac{1}{e^x - 1}$ as indicated by experiment.

The modern interpretation of Planck's radical assumption is that E&M waves come in packets of energy, "photons" γ. Thus waves exhibit particle-like behaviour.

This was interpretation was cemented by Einstein's explanation of the photoelectric effect, and the explanation of Compton scattering in terms of point-particle kinematics.