L20-Scattering and Transfer Matrices

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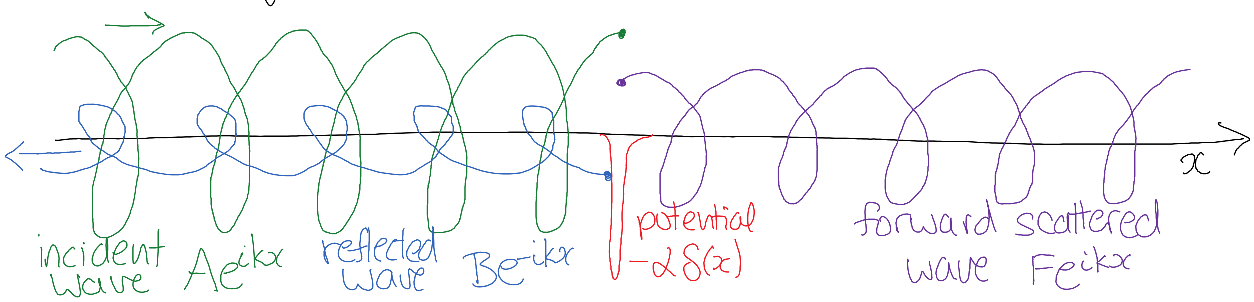
* scattering eigenstates of the δ-function potential

If $E>0$, the states will be oscillatory as $x \to \pm\infty$.
We cannot solve for discrete energy eigenstates using external B.C.'s.
There a continuous energy spectrum (the continuum):
There is a wavefunction for every energy level.
Our attention shifts from solving for energy levels
to solving for scattering (reflection)/transition amplitudes.



+Eigenfunctions: solve in two regions: $\psi_1(x)$ $[x<0]$, $\psi_2(x)$ $[x>0]$

if $x\neq 0$, $\frac{d}{dx}e^{\pm ikx} = \underbrace{\pm ik}_{\text{eigenvalue}}\, e^{\pm ikx}$ $\quad E = +\frac{\hbar^2k^2}{2m}$

general solution: $\psi_1(x) = \underset{\rightarrow}{Ae^{ikx}} + \underset{\leftarrow}{Be^{-ikx}}$ $\quad \psi_2(x) = \underset{\rightarrow}{Fe^{ikx}} + \underset{\leftarrow}{Ge^{-ikx}}$

Note: $e^{\pm ikx}$ have definite momentum, while $\sin(kx) \pm i\cos(kx)$ are easier to satisfy external boundary conditions.

+These solutions are not normalizable – continuous spectrum thus k is a free parameter and the 4 boundary conditions will give us the 4 parameters A, B, F, G.

* Boundary conditions:

+External boundary conditions are different for scattering, and represent the incoming flux from the left (A) or right (G).

The flux (particles/s) is given by the probability current at $x = \pm\infty$:

$$j_C(x) = \psi^* \frac{\hat{p}}{2m}\psi - \psi\frac{\hat{p}}{2m}\psi^* = \left(Ce^{\pm ikx}\right)^* \frac{-i\hbar\partial_x}{2m}\left(Ce^{\pm ikx}\right) + \text{c.c. [complex conjugate]}$$

$$= C^* e^{\mp ikx} \cdot \frac{-i\hbar}{2m}(\pm ikx)\, C e^{ikx} + \text{same} = \pm\frac{\hbar k}{m}|C|^2 = \pm v|C|^2$$

Thus the external boundary conditions are given specifying

$j_A(-\infty) = |A|^2 v$ and $j_G(\infty) = |G|^2 v$ [usually $(A,0)$ or $(0,G)$].

+The internal (continuity B.C.'s) are used to solve for the scattering (reflection) amplitude (B/A) or transmission amplitude (F/A): They are the same conditions as for the bound state problem:

i) $\Delta\psi = 0 \quad [\psi_2(0) = \psi_1(0)]$: $\quad F + G = A + B \quad$ [1]

ii) $\frac{-\hbar^2}{2m}\Delta\psi'(0) - \alpha\psi(0) = 0$; $\quad \frac{-\hbar^2}{2m} ik\big((F-G)-(A-B)\big) = \alpha(A+B)$

let $\beta = \frac{m\alpha}{\hbar k}$ then $\quad F - G = (1+2i\beta)A - (1-2i\beta)B \quad$ [2]

+Now apply the external boundary conditions and solve:

$$\begin{array}{rll} F &= A + B & (1) \\ F &= (1+2i\beta)A - (1-2i\beta)B & (2) \\ \hline (-) & & \\ 0 &= [1-(1+2i\beta)]A + [1-(1-2i\beta)]B & \end{array}$$

$$\begin{array}{rl} (1-2i\beta)F &= (1-2i\beta)A + (1-2i\beta)B \\ F &= (1+2i\beta)A - (1-2i\beta)B \\ \hline (+) & \\ [(1-2i\beta)+1]F &= [(1-2i\beta)+(1+2i\beta)]A \end{array}$$

Outgoing amplitude ratios: $B = \frac{i\beta}{1-i\beta}A \qquad F = \frac{1}{1-i\beta}A$

Reflection/Transmission coefficients:

$$R = \frac{j_B}{j_A} = \frac{\hbar k|B|^2}{\hbar k|A|^2} = \left|\frac{B}{A}\right|^2 = \left|\frac{i\beta}{1-i\beta}\right|^2 = \frac{\beta^2}{1+\beta^2}$$

$$T = \frac{j_F}{j_A} = \frac{\hbar k|F|^2}{\hbar k|A|^2} = \left|\frac{F}{A}\right|^2 = \left|\frac{1}{1-i\beta}\right|^2 = \frac{1}{1+\beta^2}$$

note that $R + T = \frac{1+\beta^2}{1+\beta^2} = 1$ No particle is lost.

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combine these two eq's. into single matrix eq: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$ [1] [2]

there are 4 unknowns and only 2 equations: we must solve for 2 coefficients in terms of 2 others.

which 2 you solve for depends on the application:

a) transfer matrix: $A,B \to F,G$: useful for "transferring" the amplitudes across multiple boundaries: $A,B \to C,D \to F,G \to \ldots$

b) scattering matrix: $A,G \to B,F$ – physical coefficients for reflection & transmission from the left or right

See Griffiths 1st ed. §2.7; 2nd ed problems 2.52, 2.53

+ transfer matrix: $\begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix}$ $\quad M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix} = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta & 1-i\beta \end{pmatrix}$

for multiple scatters, $M_{tot} = M_1 \cdot M_2$ PHY 417, HW5 #2

(recall recursive solution of cylindrical magnetic shields!)

(also recall reflection/transmission coefficients in E&M)

+ scattering matrix: $\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix}$ $\quad S = \frac{1}{M_{22}}\begin{pmatrix} -M_{21} & 1 \\ |M| & M_{12} \end{pmatrix} = \frac{1}{1-i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}$

Coefficients of:	reflection	transmission
from: left ($G=0$)	$R_\ell \equiv \left.\frac{\|B\|^2}{\|A\|^2}\right\|_{G=0} = \|S_{11}\|^2$	$T_\ell \equiv \left.\frac{k_2}{k_1}\frac{\|F\|^2}{\|A\|^2}\right\|_{G=0} = \frac{k_2}{k_1}\|S_{21}\|^2$
right ($A=0$)	$R_r \equiv \left.\frac{\|F\|^2}{\|G\|^2}\right\|_{A=0} = \|S_{22}\|^2$	$T_r = \left.\frac{k_2}{k_1}\frac{\|B\|^2}{\|G\|^2}\right\|_{A=0} = \frac{k_2}{k_1}\|S_{12}\|^2$