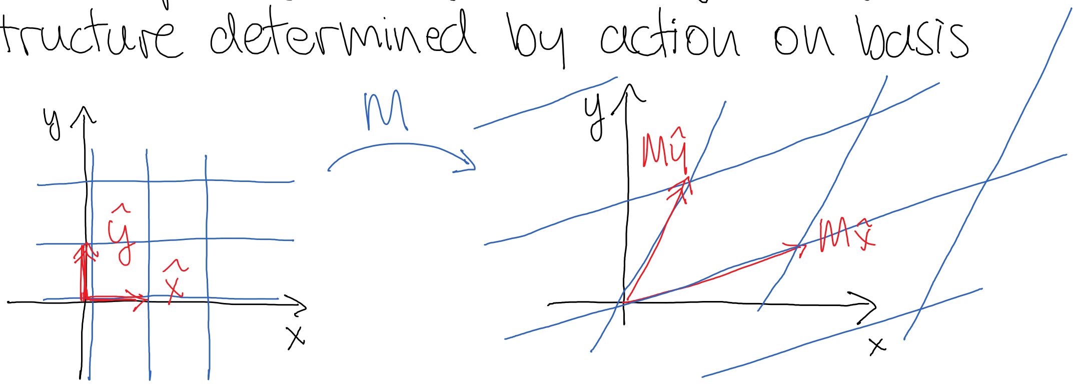
L24-Operators: Rotations

Monday, September 21, 2015 6:35 AM

* linear operators – functions from $V \to V$
  - structure determined by action on basis



* components

$$\vec{w} = M(\vec{v}) = M(\vec{b}\,\mathbb{v})$$

$$\vec{b}\,\mathbb{w} = M(\vec{b})\,\mathbb{v} = \vec{b}\,\mathbb{M}\,\mathbb{v}$$

$$\mathbb{W} = \mathbb{M}\,\mathbb{v} \quad \text{where } \boxed{\mathbb{M} = \tilde{b} \cdot M(\vec{b})}$$

closure

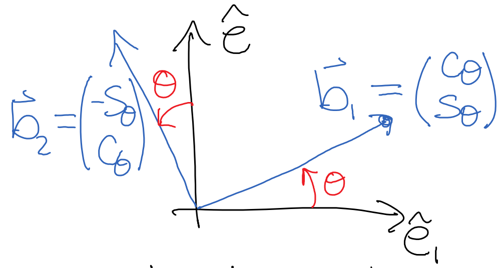
$$\underbrace{\tilde{b}(\vec{w})}_{\mathbb{W}} = \underbrace{\tilde{b}(M(\vec{b}))}_{\mathbb{M}}\,\underbrace{\tilde{b}(\vec{v})}_{\mathbb{V}}$$

$$M_{ij} = \hat{e}_i \cdot M(\hat{e}_j) \sim \langle x | A | y \rangle$$

(matrix elements)

matrix operators inherit 2 bases:
from domain & range!

* Rotations (active)



let $\vec{b}_1 = R\,\hat{e}_1 \qquad \vec{b}_2 = R\,\hat{e}_2$

combine: $(\vec{b}_1\ \vec{b}_2) = R(\hat{e}_1\ \hat{e}_2)$

$$\vec{b} = R\,\hat{e}$$

$$\begin{pmatrix} b_1^x & b_2^x \\ b_1^y & b_2^y \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

– rotations are "orthogonal" (preserve the metric)

$$\vec{b}^T \cdot \vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix} \cdot (\vec{b}_1\ \vec{b}_2) = \mathbb{I} \qquad R^T R = \mathbb{I}$$

$$\vec{b}_i \cdot \vec{b}_j = \delta_{ij}$$

in general, $\mathbb{R}^{adj} \cdot \mathbb{R} = \mathbb{I}$, or $\boxed{\mathbb{R}^\dagger G\,\mathbb{R} = G}$

* Hermitian adjoint of a operator (complex transpose)
- how an operator commutes with the metric

$$M(\vec a)\cdot\vec b \equiv \vec a\cdot M^{adj}(\vec b) \qquad \langle Mf|g\rangle = \langle f|M^\dagger g\rangle$$

$$(M\,a)^\dagger G\, b = a^\dagger G\, M^{adj}\, b$$

$$M^\dagger G = G\, M^{adj} \qquad \boxed{M^{adj} = G^{-1} M^\dagger G} \rightarrow M^\dagger \text{ if } G = I$$

- a matrix multiplies "to the left" as its adjoint

if $b = aM$ $\quad (b_1\; b_2) = (a_1\; a_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix} \quad \begin{pmatrix} b_1 \\ b_2\end{pmatrix}^* = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22}\end{pmatrix}^* \begin{pmatrix} a_1 \\ a_2\end{pmatrix}^*$

then $b^\dagger = M^\dagger a^\dagger$

note the transpose

- Adjoint is a duality: $(M^{adj})^{adj} = M$
- Adjoint of product: $(A\cdot B)^{adj} = B^{adj}\cdot A^{adj}$ like $(AB)^{-1} = B^{-1}A^{-1}$
- Unified operation (†) for adjoint of operators, vectors, scalars!

* Active vs. Passive rotations
- Active rotations: physically move vectors
  - only one basis, $\vec b_i$ are just rotated vectors.

$$\vec v' = R\vec v \qquad \vec v = \hat x v_x + \hat y v_y \rightarrow \vec v' = \hat x v_x' + \hat y v_y'$$

  - example: evolution of the wavefunction in time

- Passive rotations (identity transform) $\vec v$ stays put
  - turn your head and look at it from different angle.
  - new components correspond to a change of basis

$$\vec V = \overbrace{\vec b}\,\underbrace{v'} = \overbrace{\hat e R}\,\underbrace{v} = \hat e\, v = \vec V \text{ (still)}$$

$$\vec b = \hat e R \quad\Rightarrow\quad v' = R\, v$$

closure

$$\vec V = 1\,\vec V \qquad \underbrace{\vec b^T\cdot\vec V}_{v'} = \underbrace{\vec b^T\cdot 1\,\hat e}_{R}\,\underbrace{\hat e^T\cdot\vec V}_{v}$$

identity transform

(wait for L13)

- example: Fourier transform: $\langle x|f\rangle = \int dk \langle x|k\rangle\langle k|f\rangle$

– Similarity transformation: change of basis of a matrix

if $\mathbb{M} = \hat{e}^T \cdot M(\hat{e})$ are matrix elements in the basis $\hat{e}$,

then $\mathbb{M}' = \vec{b}^{-1} \cdot M(\vec{b}) = \underbrace{\vec{b}^{-1} \cdot \hat{e}}_{\mathbb{B}^{-1}} \underbrace{\hat{e}^T \cdot M(\hat{e}}_{\mathbb{M}} \underbrace{\hat{e}^T \cdot \vec{b}}_{\mathbb{B}})$ in the basis $\vec{b}$

thus $\boxed{\mathbb{M}' = \mathbb{B}^{-1} \mathbb{M} \mathbb{B}}$ change-of-basis formula for matrices

$\mathbb{B} = \hat{e}^T \cdot 1 \vec{b}$ transforms the domain $(\vec{v})$

$\mathbb{B}^{-1} = \vec{b}^{-1} \cdot 1 \hat{e}$ transforms the range $(\vec{w})$