L26-Adjoint, Unitarity, and Closure

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* Dirac notation for operators on Hilbert spaces

- important "tools" for going back and forth:

adjoint: $\langle f|^\dagger \equiv |f\rangle \quad z^\dagger \equiv z^* \quad \langle f|H^\dagger|g\rangle = \langle g|H|f\rangle^*$

orthonormality: $\langle n|m\rangle = \delta_{nm} \quad \langle x|x'\rangle = \delta(x-x')$

closure: $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \quad \int dx\, |x\rangle\langle x| = 1$

components: $\langle n|f\rangle = f_n \quad \langle x|f\rangle = f(x) \quad \langle x|n\rangle = \phi_n(x)$

matrix elements: $\langle m|H|n\rangle = H_{mn} \quad \langle x|H|x'\rangle = H(x-x')$

- what is the "matrix" of an ∞-d operator?

$$|g\rangle = H \quad |f\rangle$$
$$\langle m|g\rangle = \langle m|H \sum_n |n\rangle\langle n| \, |f\rangle$$
$$g_m = \sum_n H_{mn} f_n$$

$$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \cdots \\ H_{21} & H_{22} & \\ \vdots & & \ddots_{\infty} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

$$\langle x|g\rangle = \langle x| H \int dx' |x'\rangle\langle x'| \, |f\rangle$$
$$g(x) = \int dx' \underbrace{H(x,x')}_{\text{kernel of integral transform}} f(x')$$

- example: identity transform $1|f\rangle = |f\rangle$

$$\langle x|f\rangle = \int dx' \langle x|1|x'\rangle \langle x'|f\rangle$$
$$f(x) = \int \delta(x-x')\, f(x')$$

* what does matrix look like?

$$\langle x|f\rangle = \int dk \langle x|1|k\rangle \langle k|f\rangle$$
$$\tilde{f}(k) = \int dx\, e^{ikx} f(x)$$

* matrix?

- example: derivative operator

* matrix?

$$f'(x) = \int \delta(x-x') \frac{d}{dx'} f(x')\, dx' = -\int f(x') \frac{d}{dx'} \delta(x-x')\, dx'$$

so $\langle x | \frac{d}{dx} | x' \rangle = -\delta'(x-x')$

* Hermitian (symmetric) operators: spectrum

- any matrix has n complex eigenvalues (FTA)
- some "defective" matrices have fewer eigenvectors

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \rightarrow J = \begin{pmatrix} \lambda_1 & 1 & & & \\ 0 & \lambda_1 & & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & \\ & & & & \ddots \end{pmatrix}$$

Jordan blocks

- Hermitian matrices $H^\dagger = H$ are Hermitian in any orthogonal basis: if $M = U^\dagger H U$ then $M^\dagger = (U^\dagger H U)^\dagger = U^\dagger H^\dagger U^{\dagger\dagger} = U^\dagger H U = M$

① Thus Hermitian matrices are NOT defective

let $H\vec{u}_i = \lambda_i \vec{u}_i$ then $\vec{u}_i^\dagger H^\dagger = \lambda_i^* \vec{u}_i^\dagger$

$$\lambda_j^* \vec{u}_j^\dagger \vec{u}_i = \vec{u}_j^\dagger H \vec{u}_i = \vec{u}_j^\dagger \vec{u}_i \lambda_i$$

② If $i = j$, $\lambda_i^* = \lambda_i$ H has real eigenvalues!

If $\lambda_i = \lambda_j$ then any $\alpha \vec{u}_i + \beta \vec{u}_j$ is an eigenvector

③ If $\lambda_i \neq \lambda_j$, $\vec{u}_j^\dagger u_i = 0$ H has orthogonal eigenspaces!

* Example: the operator $D \sim \frac{d}{dx}$ is antiHermitian

$$\langle g | D | f \rangle \equiv \int dx\, g(x) \frac{d}{dx} f(x) = \int g\, df$$

$$= \cancel{fg\big|} - \int f\, dg = -\langle f|D|g\rangle$$

somehow!

Thus the operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ is Hermitian.

* Application: Sturm Liouville theory of ODE's

Given a 2nd order differential operator L

$$L[y] = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)$$

there exists an integrating function

$$w(x) = \frac{1}{p_2(x)}\exp\left(\int_0^x \frac{p_2(x')}{p_1(x')}dx'\right)$$ such that

$$L = \frac{1}{w(x)}\left(\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right)$$ is self-adjoint $L^\dagger = L$

with respect to the inner product

$$\langle y_2|y_1\rangle \equiv \int_a^b w(x)\,dx\; y_2^*(x)\, y_1(x)$$

if $\quad w(x)\, y_2^*(x)\, y_1(x)\Big|_a^b = 0 \quad$ (boundary conditions)

Thus $L = \sum_{n=1}^{\infty} \lambda_n |n\rangle\langle n|$ has a complete

set of orthogonal eigenfunctions, since

$$\lambda_2^* \langle y_2|y_1\rangle = \langle y_2|L|y_1\rangle = \langle y_2|y_1\rangle\lambda_1$$

which is the same as above for matrix H.